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TITLE OF THESIS A FAMILY OF OPERATIONAL CALCULI

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A FAMILY OF OPERATIONAL CALCULI

by

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A THESIS

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research, for acceptance, a thesis entitled A FAMILY OF OPERATIONAL CALCULI submitted by MOURAD EL-HOUSSIENY ISMAIL in partial fulfilment of the requirements for the degree of Doctor of Philosophy.

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ABSTRACT

In a series of articles by Ditkin and Prudnikov, Meller, Osipov and others, an operational calculus for the operator DxD, and later for $x^{-\gamma}Dx^{\gamma+1}D$, was developed and several interesting consequences and applications were pointed out. It also became apparent that the Bessel operator $B = DxD$ plays the role of a derivative on the convolution product

$$(f*g)(x) = \frac{d}{dx} \int_{o}^{x} f(x-t)g(t)dt \quad ,$$

in the sense that

$$B(f*g) = (Bf)*g + f*(Bg) \quad .$$

This thesis starts by studying operational calculi for which $B_1 = \Delta x\nabla$, or in general $B_2 = \Delta(x+\gamma)\nabla$, play the role of differentiation. In each case a convolution $*$ was constructed such that

$$B_i(f*g) = (B_if)*g + f*(B_ig), \quad i = 1,2, \quad .$$

A transform L was also introduced in each case that satisfies $L(f*g) = LfLg$ and $LB_if = DLf$ for $i = 1,2$. The first operational calculus is essentially due to L. Berg but we proved more results in that direction and included some novel applications. In case of B_1 and B_2 it was shown that any linear difference equation with polynomial coefficients is transformed under L to a linear differential equation with polynomial coefficients. This analogy provides a method of solving one of them by knowing the solution to the other. Moreover, in studying the first couple of operational calculi, we were led to

study transforms of the type $Lf = \sum_{0}^{\infty} \frac{x^n}{(x+1)^{n+1}} f(n)$ and $Lf = \sum \frac{x^n}{(x+1)^{n+1+\gamma}} f(n)$, which are special cases of Jakimovski's sequence to function transform, the $[J,f(x)]$ transform. Operational calculi were developed. We also included some q finite difference analogues.

Later in the thesis we studied expansions of powers of the discrete Bessel operator $\Delta x \nabla$; or more generally powers of any operator Γ that satisfies $\Gamma\binom{x}{n} = c_n \binom{x}{n-1}$, $c_0 = 0$, $c_n \neq 0$ for $n = 1,\ldots;$ in powers of the shift operator E and its inverse.

We also introduce a discrete analogue of orthogonality with respect to convolution and used it to show that the zeros of the Rice polynomials are real and simple.

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CHAPTER I

INTRODUCTION

Mikusinski developed his operational calculus via an algebraic approach. He embedded the integral domain $(C[0,\infty),+,*)$, where

$$(1.1) \qquad f*g(x) = \int_0^x f(u)g(x-u)\,du \quad ,$$

into its field of quotients. This quotient field is rich enough to contain the integral operator $\int_0^x f(t)\,dt$, $f \in C[0,\infty)$, and hence its inverse, the differentiation operator $\frac{d}{dt}$. In this setting, a differential equation is nothing but an algebraic equation in the field of quotients. Solving this equation means finding a convolution quotient that satisfies it and if this convolution quotient is a function, then it will be the solution of the differential equation.

Mikusinski's approach has become a standard approach and several operational calculi were developed by using different convolution products. Rajewski replaced (1.1) by

$$(1.2) \qquad f*g(x) = \frac{d}{dx} \int_0^x f(u)g(x-u)\,du \ .$$

The resulting ring $(C[0,\infty),+,*)$ contains a unit element, the constant function 1. It is clear that Mikusinski's ring does not have a unit element. Note that Rajewski's convolution also preserves dimensions. If we use the Carson-Laplace transform

$$(1.3) \qquad L[f;x] = x^{-1} \int_0^\infty e^{-t/x} f(t)\,dt \quad ,$$

then

$$(1.4) \qquad L[f*g;x] = L[f;x]L[g;x] \quad ,$$

where the convolution in (1.4) is that of (1.2). Let us denote the operator $\frac{d}{dx} x \frac{d}{dx}$ by B. The operator B was called Bessel operator by Ditkin and Prudnikov. B acts on the convolution (1.2) as a differentiation. Indeed

$$(1.5) \qquad B(f*g) = f*(Bg)+(Bf)*g \quad .$$

The obvious relation

$$B \frac{x^n}{n!} = n \frac{x^{n-1}}{(n-1)!}$$

shows that $\frac{x^n}{n!}$ in the B calculus is the analogue of x^n in the differential calculus. Ditkin and Prudnikov [13] introduced an operational claculus for the Bessel operator. This calculus enabled them to treat any linear equation in B with constant coefficients as an algebraic equation in a certain field. Their convolution is

$$(1.6) \qquad F*G(x) = \frac{d}{dx} x \frac{d}{dx} \int_0^x \int_0^1 F(\xi\eta)G[(t-\xi)(1-\eta)]d\xi d\eta \quad .$$

Under this convolution, we have

$$\frac{x^k}{(k!)^2} * \frac{x^j}{(j!)^2} = \frac{x^{k+j}}{((k+j)!)^2} \quad .$$

Meller [21] developed an operational calculus for the generalized Bessel operator $B_\alpha = x^{-\alpha} \frac{d}{dx} x^{\alpha+1} \frac{d}{dx}$, which contains Ditkin and Prudnikov's as the special case $\alpha = 0$.

Amerbaev and Naurzbaev [2] introduced the notion of orthogonality with respect to convolution. The idea is to represent a known polynomial set $P_n(x)$, orthogonal on $[0,\infty)$, as the Carson-Laplace transform of another polynomial set $Q_n(x)$, that is

$$(1.7)\qquad P_n(x) = x^{-1}\int_0^\infty e^{-t/x}\, Q_n(t)\,dt, \quad x > 0 .$$

Substituting for $P_n(x)$, from (1.7) in the orthogonality relation for $P_n(x)$ we end up with a kind of orthogonality involving the Q_n's. Amerbaev and Naurzbaev [2] studied properties of the Q_n's of (1.7) when $P_n(x)$ are the Jacobi polynomials with argument $(1-x)/2$. Let $Q_n^{(\alpha,\beta)}(x)$ be the corresponding polynomials. The convolution orthogonality relation is

$$\int_0^\infty \frac{e^{-t}}{t}\frac{d}{dt}\Big\{\int_0^t\int_0^{t-\tau_2} \frac{(t-\tau_1-\tau_2)^\beta}{\Gamma(\beta+1)}\, {}_1F_1(-\alpha;\beta+1;t-\tau_1-\tau_2)$$

$$\times\, Q_n^{(\alpha,\beta)}(\tau_1)Q_n^{(\alpha,\beta)}(\tau_2)\,d\tau_1 d\tau_2\Big\} = \frac{\Gamma(\alpha+n+1)\Gamma(n+\beta+1)}{n!\,(2n+\alpha+\beta)\Gamma(\alpha+\beta+n+1)}\,\delta_{m,n} .$$

They utilized (1.5) to deduce several properties of the polynomials $Q_n^{(\alpha,\beta)}(x)$ from those of $P_n^{(\alpha,\beta)}(x)$. Among other things they obtained differential equations, recurrence relations and generating functions for $Q_n^{(\alpha,\beta)}(x)$. These polynomials are not orthogonal, in the usual sense, on any subset of the real line and it is this fact that makes this new concept interesting.

Berge [8] introduced an operational calculus on sequences using the convolution

$$f*g(n) = \sum_{k=o}^{n} \binom{n}{k} f(k)g(n-k) \quad ,$$

and used the exponential generating function as his transform. Moore [26], Brand [10] and Traub [36] studied an operational calculus on sequences using the product

$$f*g(n) = \sum_{o}^{n} f(k)g(n-k) \quad ,$$

which is a discrete analogue of Mikusinki's convolution (1.1).

In [6] Berg developed an operational calculus for sequences based on the convolution

$$f*g = \sum_{o}^{n} f_k \, g_{n-k} - \sum_{o}^{n-1} f_k \, g_{n-k-1} \quad . \tag{1.8}$$

Berg's convolution in the discrete analogue of (1.2).

In this thesis we give several discrete analogues of the Bessel operator and its generalization and construct operational calculi for them.

In Chapter II we consider the Berg operational calculus in more detail than was done before obtaining new results as well as indicating some novel applications. We shall find out here that a discrete analogue for $D \times D$ is $\Delta \; n \; \nabla$. We then generalize Berg's calculus to a discrete operational calculus which corresponds to the generalized Bessel operator $x^{-\alpha} D x^{\alpha+1} D$. We note here that the operations which map the Bessel operators to the derivatives (analogue of Carson-Laplace transforms) are special cases of Jakimowski's sequence to function transform known as the $[J,f(x)]$

means [19]. In Chapter III we generalize the results of Chapter II by showing that a large class of Jakimowski type means induce corresponding operational calculi by defining for each member of this class a convolution product, a Bessel-type operator and a Carson-Laplace-type transform preserving the basic properties of Chapter II.

In Chapter IV we consider a discrete analogue of orthogonality with respect to convolution. We use our concept to obtain the apparently unknown result that the zeros of the Rice polynomials [32] are real and simple.

Finally in Chapter V we derive expansion formulas for linear operators Γ satisfying

$$\Gamma\binom{x}{j} = c_j\binom{x}{j-1} ,$$

where $c_0, c_1, \ldots$ is an arbitrary sequence with $c_0 = 0$. The discrete Bessel operator β is such a Γ operator with $c_j = j$. Later, in Chapter V we develop, very briefly, some q-analogues of the (L,α) transforms and operational calculi. We also outline a possible way of extending the (L,α) transforms to several variables.

CHAPTER II

BERG'S OPERATIONAL CALCULUS

2.1 It is well known that the action of the Bessel operator $B = \frac{d}{dt} t \frac{d}{dt}$ on Rajewski's convolution product

$$f*g(t) = \frac{d}{dt} \int_o^t f(u)g(t-u)du \quad ,$$

resembles the action of the differentiation operator $\frac{d}{dt}$ on ordinary products. The image of a convolution product under the Carson-Laplace transform

$$L[f;x] = x^{-1} \int_o^\infty e^{-t/x} f(t)dt \qquad x > o \, ,$$

is the product of the transforms, that is

$$L[f*g;x] = L[f;x]L[g;x] \, .$$

Moreover L transforms B to $\frac{d}{dx}$, in the sense that

$$L[Bf;x] = \frac{d}{dx} L[f;x] \, ,$$

and maps $t^n/n!$ to x^n. Thus the role of $\frac{d}{dx}$ and x^n in the range of L is played by B and $t^n/n!$ in the domain of L. This observation led to some interesting results ([2], [13]), and so it is of interest to study a discrete analogue of this calculus.

In the calculus of finite differences, the sequence $\{\binom{n}{j}, \; j = 0,1,\ldots\}$ may be considered as the discrete analogue of the functions $\{ \frac{t^j}{j!}, \; j = 0,1,\ldots\}$. This is so since

$$\frac{d}{dt}\{t^j/j!\} = t^{j-1}/(j-1)! \quad \text{and} \quad \Delta_n \binom{n}{j} = \binom{n}{j-1} ,$$

where

(2.1) $$\Delta_n f(n) = f(n+1) - f(n) .$$

Because of the relation

$$B \frac{t^j}{j!} = j \frac{t^{j-1}}{(j-1)!} ,$$

we are interested in finding a finite difference operator β, say, which takes $\binom{n}{j}$ to $j\binom{n}{j-1}$. This is accomplished by

(2.2) $$\beta_t = \Delta_t \, t\nabla_t ,$$

where

(2.3) $$\nabla_t \, f(t) = f(t) - f(t-1) .$$

To see this we put, formally, $\beta_t = \sum_0^\infty a_k(t)\Delta^{k+1}$ and use the requirement $\beta_t\binom{t}{r} = r\binom{t}{r-1}$ $r = 0,1,\ldots,$ to get

$$a_o(t) = 1 \quad \text{and} \quad a_k(t) = (-1)^k t , \quad k = 1,2,\ldots,$$

so that

$$\beta = \Delta + t \sum_0^\infty (-1)^k \Delta^{k+2} = \Delta + t \frac{\Delta^2}{1+\Delta} .$$

The operator $1+\Delta$ is the shift operator E defined by

(2.4) $$E_t \, f(t) = f(t+1) .$$

Clearly,

$$\frac{\Delta}{1+\Delta} = \frac{\Delta}{E} = \nabla ,$$

hence

$$\beta_t = \Delta + t\Delta\nabla = \Delta t\nabla ,$$

which is (2.2). The assertion

(2.5) $$\beta_t\binom{t}{j} = j\binom{t}{j-1} \qquad j = 0,1,\ldots \quad \text{with} \quad \binom{t}{-1} = 0 ,$$

can be easily verified. It is also easy to see that

(2.6) $$\beta_t\, f(t) = (t+1)f(t+1) - (2t+1)f(t) + tf(t-1) .$$

In the following we shall drop the subscript t from Δ_t, E_t, ∇_t or β_t whenever no confusion arises from doing so.

We introduce a discrete analogue of the Carson-Laplace transform later.

Let S be the set of all sequences $\{f(n)\}_{-\infty}^{\infty}$ with $f(-1) = f(-2) = \ldots = 0$. S can be made into a complex vector space, in a natural way, by defining the addition of two elements of S by componentwise addition and scalar multiplication by $\alpha f = \{\alpha f(n)\}$ for any complex number α.

Throughout this thesis, for $f \in S$, by $f(t)$ we mean the t^{th} component of f and formulas like (2.1), (2.2) and (2.3) should be interpreted as the sequence, in S, whose t^{th} component is what appears on the right hand side.

Now that we have found β, the next step is to look for an appropriate convolution product that is well defined on S and satisfies

(2.7) $$\beta(f*g) = f*\beta g + \beta f*g \quad .$$

The convolution

(2.8) $$f*g(n) = \nabla \sum_{0}^{n} f(k)g(n-k) \quad ,$$

or equivalently

(2.9) $$f*g(n) = \sum_{0}^{n} f(k)g(n-k) - \sum_{0}^{n-1} f(k)g(n-k-1) \quad ,$$

is a natural difference analogue to Rajewski's convolution $\frac{d}{dt}\int_{0}^{t} f(u)g(t-u)$, hence is a candidate for the convolution sought. Indeed we have

Theorem 2.1. The convolution product (2.8) satisfies (2.7).

Proof. Clearly we have

$$f*g(n) = \Delta \sum_{0}^{n-1} f(k)g(n-k-1) \quad .$$

Thus

(2.10) $$(\beta f*g+f*\beta g)(n) = \Delta \sum_{0}^{n-1} \{g(n-k-1)\beta f(k)+f(k)\beta g(n-k-1)\} \quad .$$

Substituting for βf and βg from (2.6) in the right hand side of (2.10) and simplifying the resulting expression we get

$$\begin{aligned}(\beta f*g+f*\beta g)(n) &= \Delta n \{ \sum_{0}^{n} f(k)g(n-k) - 2 \sum_{0}^{n-1} f(k)g(n-k-1) \\ &\qquad + \sum_{0}^{n-2} f(k)g(n-k-2)\} \\ &= \beta(f*g)(n) \quad .\end{aligned}$$

L. Berg [6], [7] has constructed, a la Mikusinski, an operational calculus based on the convolution (2.8). In the next section we mention few properties of that operational calculus and derive several properties that are not contained in [6], [7]. We shall refer to this particular operational calculus as Berg's Operational Calculus. For similar type of results see [14]. Note that discrete operational caculi based on different convolutions have been studied in [8], [10], [25], [26] and [27].

2.2 Berg's Operational Calculus. It is obvious that $(S, +, *)$ is a commutative ring without zero divisors. For brevity, we shall denote this ring by S also. Let $\{Y, +, *\}$, or simply Y, be its field of quotients. The following formulas may be found in [7].

$$\{1,0,0,\ldots\}*\{f(n)\} = \{\nabla f(n)\} \tag{2.11}$$

$$\{n\}*\{\Delta f(n)\} = \{f(n)-f(0)\} \tag{2.12}$$

$$\{n+1\}*\{f(n)\} = \{\sigma f(n)\} , \tag{2.13}$$

where

$$\sigma f(n) = \sum_{o}^{n} f(k) . \tag{2.14}$$

Let

$$\alpha_k(t) = \binom{t}{k} , \qquad k = 0,1,\ldots . \tag{2.15}$$

The fundamental relation

$$\alpha_k(t)*\alpha_j(t) = \alpha_{k+j}(t) , \qquad k,j = 0,1,\ldots , \tag{2.16}$$

also appears in [7]. Relations (2.11) and (2.13) provide representations for ∇ and σ in the field Y. The relation (2.16) expresses the fact that the functions $\alpha_o(t)$, $\alpha_1(t)$, $\alpha_2(t)$, ... can be treated like powers 1, x, x^2, ... Formula (2.12) shows that Δ may be identified with the convolution quotient $\frac{\{1\}}{\{n\}}$ on the manifold $\{f:f \in S$ and $f(0) = 0\}$. Let $\theta\{f(n)\} = \{nf(n)\}$. It is easy to see that the operators β and θ are not members of Y in the sense that there is no convolution quotient $\frac{a}{b} \in Y$ that satisifes $\theta f = \frac{a}{b} * f$ or $\beta f = \frac{a}{b} * f$.

<u>Theorem 2.2</u>. <u>For</u> $f \in S$ <u>we have</u>

$$\{n \nabla f(n)\} = \alpha_1 * \beta f \quad , \tag{2.17}$$

and

$$\{nf(n-1)\} = \alpha_2 * \beta f + \alpha_1 * f \quad . \tag{2.18}$$

<u>Proof</u>. By (2.13) we have

$$\alpha_1 * \beta f = \{ \sum_o^{n-1} \beta f(k)\} = \{ \sum_o^{n-1} \Delta k \nabla f(k)\} = \{n\nabla f(n)\} ,$$

proving (2.17). Now

$$\alpha_2 * \beta f = \alpha_1 * \alpha_1 * \beta f = \alpha_1 * \{n \nabla f(n)\} \quad ,$$

by (2.17), so that

$$(\alpha_2 * \beta f)(n) = \sum_o^{n-1} k\{f(k)-f(k-1)\} = \sum_o^{n-1} kf(k) - \sum_o^{n-2} (k+1)f(k)$$

$$= (n-1)f(n-1) - \sum_o^{n-2} f(k) = nf(n-1) - (\alpha_1 * f)(n) \quad .$$

Therefore

$$(\alpha_2*\beta f+\alpha_1*f)(n) = n\ f(n-1) \quad ,$$

and (2.18) follows.

Corollary.

$$\{nf(n)\} = (\alpha_1+\alpha_2)*\beta f + \alpha_1*f \quad . \tag{2.19}$$

For the definition of convergence of convolution quotients we refer the reader to [7] and [16]. In this regard we have

Theorem 2.3. *If* $f = \{f(0), f(1), \ldots\} \in S$, $f \neq \{1,1,1,\ldots\}$, *then in order that* $\lim_{n \to \infty} f^{(n)}$ *exists, where* $f^{(n)} = \underbrace{f*f*\ldots*f}_{n\text{-times}}$, *it is necessary and sufficient that* $|f(0)| < 1$. *Moreover, if this limit exists, it will be zero.*

The proof will be postponed until the end of the next section because it uses some results to be developed there.

2.3 A Discrete Analogue of Carson-Laplace Transform. Now we come to the problem of finding a discrete analogue of the Carson-Laplace transform. Denote the required transform by L. We require L to be linear, to map S into the ring of formal power series and to have a nontrivial range. We write $L[f;x]$ to mean the image under L of $f(\epsilon S)$. L must also satisfy

$$L[\beta f;x] = \frac{d}{dx} L[f;x] \tag{2.20}$$

and

$$L[f*g;x] = L[f;x]L[g;x] \quad . \tag{2.21}$$

The linearity of L implies that it must be of the form

$$(2.22) \qquad L[f;x] = \sum_{0}^{\infty} a_k(x)f(k) \quad ,$$

where $a_o(x)$, $a_1(x)$, ... are power series in x.

<u>Theorem 2.4. The transform</u> L <u>is given by</u>

$$(2.23) \qquad L[f;x] = \sum_{0}^{\infty} \frac{x^n}{(1+x)^{n+1}} f(n) \quad ,$$

<u>up to a shift in</u> x.

<u>Proof.</u> Set

$$\theta_k(x) = L[\alpha_k;x] \quad , \qquad k = 0,1,\ldots \quad .$$

The relation $\alpha_o f = f$ for all $f \in S$ implies, by (2.21),

$$L[\alpha_o;x]L[f;x] = L[f;x] \quad ,$$

and hence

$$\theta_o = L[\alpha_o;x] = 1 \; .$$

Using (2.20) we get $\theta_1'(x) = \theta_o(x)$, that is

$$\theta_1(x) = x+C, \qquad C \text{ is constant.}$$

Recall that $\alpha_k = \underbrace{\alpha_1 * \alpha_1 * \ldots * \alpha_1}_{k\text{-times}}$, for $k > 0$. Thus by (2.21) and the definition of $\theta_k(x)$ we have

$$\theta_k(x) = \{L[\alpha_1;x]\}^k = (x+c)^k \quad .$$

Thus

$$(x+c)^j = \sum_{k=o}^{\infty} a_k(x)\binom{k}{j} \quad .$$

Multiply both sides of the above equation by $(-1)^{j-n}\binom{j}{n}$ and summing over all j we get

$$(2.24) \qquad \frac{(x+c)^n}{(1+x+c)^{n+1}} = \sum_{k=o}^{\infty} a_k(x) \sum_{j=o}^{\infty} \binom{k}{j}\binom{j}{n}(-1)^{j+n} \quad ,$$

and by virtue of the well-known identity

$$\sum_{j=o}^{\infty} (-1)^{j+n} \binom{k}{j}\binom{j}{n} = \delta_{j,n} \quad ,$$

this formula reduces to

$$a_n(x) = (x+c)^n/(1+x+c)^{n+1} \quad ,$$

and we take $c = o$ since, in general, we allow a shift in x. Now it only remains to show that (2.20) and (2.21) are satisfied for $f,g \in S$.

$$L[f*g;x] = \sum_{n=o}^{\infty} \frac{x^n}{(1+x)^{n+1}} \left\{ \sum_{k=o}^{n} f(k)g(n-k) - \sum_{o}^{n-1} f(k)g(n-k-1) \right\}$$

$$= \sum_{o}^{\infty} f(k)\left(\frac{x}{1+x}\right)^k \sum_{n=o}^{\infty} \frac{x^n}{(1+x)^{n+1}} g(n)$$

$$- \sum_{k=o}^{\infty} f(k)\left(\frac{x}{x+1}\right)^{k+1} \sum_{o}^{\infty} \frac{x^n g(n)}{(1+x)^{n+1}}$$

$$= L[f;x]L[g;x] \quad .$$

Finally we have

$$L[\beta f;x] = \sum_{0}^{\infty} \frac{x^n}{(x+1)^{n+1}} \{(n+1)f(n+1) - (2n+1)f(n) + nf(n-1)\}$$

$$= \frac{x+1}{x} \sum_{0}^{\infty} \frac{n\,x^n}{(1+x)^{n+1}} f(n) - 2\sum_{0}^{\infty} \frac{x^n}{(1+x)^{n+1}} nf(n)$$

$$- \sum_{0}^{\infty} \frac{x^n}{(1+x)^{n+1}} f(n) + \frac{x}{1+x} \sum_{0}^{\infty} \frac{(n+1)x^n}{(1+x)^{n+1}} f(n)$$

$$= (1+x)^{-1} \sum_{0}^{\infty} \left(\frac{x}{1+x}\right)^n f(n) \{n + \frac{n}{x} - 2n-1 + (n+1)x/(x+1)\}$$

$$= (1+x)^{-1} \sum_{0}^{\infty} \left(\frac{x}{1+x}\right)^n f(n) \{\frac{n}{x} - \frac{n+1}{x+1}\} = \frac{d}{dx} L[f;x] \ ,$$

and the proof is complete.

From now on, we adopt (2.23) as the definition of the transform L. We shall always use small letters to denote members of S and capital letters to denote their transforms under L, that is $f(x)$ will stand for $L[f;x]$.

The above L transform appears in Berg [7] and Ditkin and Prudnikov [14] with a change of variable. However, no attempt was made there to derive (2.23) as we did in Theorem 2.4.

Now we look at transforms of the fundamental finite difference operators. The formulas

(2.25) $$L[nf(n);x] = xD\{(x+1)F(x)\} \ , \quad \text{where} \quad D = \frac{d}{dx} \ ,$$

(2.26) $$L[\nabla f(n);x] = F(x)/(x+1) \ ,$$

(2.27) $$L[\Delta f;x] = \{F(x) - F(0)\}/x \ ,$$

and

$$(2.28) \qquad L[\sigma f;x] = (x+1)F(x) \quad ,$$

follow easily from the definition of L, hence we omit their proofs. By induction we get

$$(2.29) \qquad L[n^k f(n);x] = \{xD(x+1)\}^k F(x) \ ,$$

$$(2.30) \qquad L[\nabla^k f(n);x] = F(x)/(1+x)^k$$

and

$$(2.31) \qquad L[\Delta^k f;x] = \{F(x)-F(0)-xF'(0)-\ldots-x^{k-1}F^{(k-1)}(0)\}/x^k \ , \quad k > 0 \ ,$$

where $F(0)$, $F'(0)$, ..., are defined by

$$F(x) = \sum_0^\infty x^k F^{(k)}(0) \quad .$$

The formulas

$$(2.32) \qquad L[(n\nabla)^k f;x] = (xD)^k F(x)$$

and

$$(2.33) \qquad L[\{(n+1)\Delta\}^k f;x] = \{(x+1)D\}^k F(x)$$

follow also easily by induction.

The formula

$$(2.34) \qquad L[n^{(k)}\nabla^k f;x] = x^k D^k F(x) \quad ,$$

where

$$n^{(k)} = \begin{cases} 1 & \text{for } k = 0 \\ n(n-1)\ldots(n-k+1) & \text{for } k > 0 \end{cases} ,$$

may be proved as follows. $k = 1$ follows by (2.32). If (2.34) is

valid for some k, then

$$L[n^{(k+1)}\nabla^{k+1}f;x] = L[(n-k)n^{(k)}\nabla^k\nabla f;x] = xD(1+x)L[n^{(k)}\nabla^k\nabla f;x]$$
$$- kL[n^{(k)}\nabla^k\nabla f;x]$$
$$= xD(1+x)x^kD^k\{F(x)/(1+x)\}-kx^kD^k\{F(x)/(1+x)\}$$
$$= (k+1)x^{k+1}D^k\{F(x)/(1+x)\}+x^{k+1}(1+x)D^{k+1}\{F(x)/(x+1)\}$$
$$= x^{k+1}D^{k+1}(x+1)\{F(x)/(x+1)\} ,$$

and the induction is complete.

Relations (2.29), (2.30) and (2.31) show that the L transform of a linear difference equation with polynomial coefficients is a linear differential equation with polynomial coefficients too. In particular, a linear equation with constant coefficients is transformed under L to an algebraic equation. Relations (2.32) and (2.33) show that difference equations of the types

$$\sum_{k=o}^{\ell} a_k\{(n+1)\nabla\}^k f(n) = g(n) \quad \text{and} \quad \sum_{k=o}^{\ell} a_k(n\nabla)^k f(n) = g(n) ,$$

are transformed to

$$\sum_{k=o}^{\ell} a_k\{(x+1)D\}^k F(x) = G(x) \quad \text{and} \quad \sum_{k=o}^{\ell} a_k(xD)^k F(x) = G(x) ,$$

respectively, when $a_o, a_1, \ldots, a_\ell$ are constants. Furthermore according to (2.34) the difference equation

$$\sum_{k=o}^{\ell} a_k\, n^{(k)}\nabla^k f(n) = g(n) ,$$

is transformed to Euler's equation

$$\sum_{k=o}^{\ell} a_k \, x^k D^k F(x) = G(x)$$

The above correspondence provides a possibly useful method of solving difference equations because the resulting differential equation is very often easier to solve than the original difference equation.

Let $X = \{f : f \in S \text{ and } \sup |f(n)|^{1/n} < \infty\}$. Thus, for $f \in X$, $F(x)$ will be holomorphic in a neighborhood of the origin. It is plain that X is closed under addition. X is also closed under convolution. For, if $n > 0$ then

$$\begin{aligned} \left|\sum_{k=o}^{n} f(k)g(n-k)\right|^{1/n} &\leq (n+1)^{1/n} \sup \{|f(k)|^{1/n},\ k = 0,1,\ldots,n\} \\ &\qquad \times \sup \{|g(k)|^{1/n},\ k = 0,1,\ldots,n\} \\ &\leq A^2 (n+1)^{1/n} \quad , \end{aligned}$$

where

$$A = \max \{1, \sup_{k>o} |f(k)|^{1/k}, \ \sup_{k>o} |g(k)|^{1/k}, \ f(0), g(0)\} .$$

Therefore $\sup_n |f*g(n)|^{1/n} < \infty$ and $f*g \in X$.

For $f \in X$, or equivalently $F(x)$ is holomorphic in a neighborhood of the origin, the inversion formula

$$f(n) = \frac{1}{2\pi i} \int_c F\left(\frac{z}{1-z}\right) z^{-n-1} (1-z)^{-1} \, dz \quad , \tag{2.35}$$

or

$$(2.36) \qquad f(n) = \frac{1}{2\pi i} \int_c \frac{F(z)}{z^{n+1}} (1+z)^n \, dz \quad ,$$

with an appropriate contour c, follow by Cauchy's theorem. Note that if $f \in X$, then so will be Δf, σf, ∇f, $\{nf(n)\}$, ... and all the operations we performed so far, for example rearranging terms in series, are justifiable from the theory of functions point of view. That L is one-to-one follows from the identity theorem for power series. Note that the proof of

$$(2.37) \qquad L[\tbinom{n}{k};x] = x^k \quad ,$$

is contained in Theorem 2.4. Relation (2.37) provides an easy inversion formula. If $F(x) = \sum_o^\infty a_k x^k$, is any formal power series, then $f(n) = \sum_o^n a_k \binom{n}{k}$.

It is also easy to see that

$$(2.38) \qquad L[f;x] = \sum_o^\infty x^n \Delta^n f(0) \ .$$

In solving difference equations one might use the operational calculus approach or the L transform technique. We would rather use the L-transform in our illustrations and examples.

Now we are in a position to prove Theorem 2.3.

Proof of Theorem 2.3. Clearly

$$(2.39) \qquad \{L[f;x]\}^k = \sum_{n=o}^\infty x^n \sum_{j_1+\dots+j_k=n} \Delta^{j_1} f(0) \Delta^{j_2} f(0) \dots \Delta^{j_k} f(0) \ .$$

Therefore $f^{(k)}(0) = \{f(0)\}^k$ for all k, and $f^{(k)}(0)$ converges, as $k \to \infty$, if and only if $|f(0)| < 1$ or $f(0) = 1$. In the latter case, $f(0) = 1$, we get

$$f^{(k)}(1) = 1+k\Delta f(0)\{f(0)\}^{k-1} = 1+k\Delta f(0) \quad .$$

This implies $\Delta f(0) = 0$ and by induction we may deduce that $\Delta^k f(0) = 0$ for $k > 0$. Therefore, in this case, $f = \{1\}$ which is a contradiction.

Conversely, let $|f(0)| < 1$. Then (2.39), for $k > n$, implies

$$|f^{(k)}(n)| \leq |f(0)|^{k-n}\binom{k}{n} \sum_{j_1+\ldots+j_n=n} |\Delta^{j_1} f(0)|\ldots|\Delta^{j_n} f(0)| \ ,$$

which shows that $f^{(k)}(n)$ converges to zero, as $k \to \infty$, and the sufficency part follows.

2.4 Examples and Applications. In our first example we solve the three term recurrence relation satisfied by the Hermite's polynomials $H_n(y)$

$$H_{n+1}(y) = 2yH_n(y) - 2nH_{n-1}(y) \ , \qquad n = 0,1,\ldots \ ,$$

with $H_{-1}(y) = 0$, $H_o(y) = 1$. The substitution $H_n(y) = \theta_n(y)\ (2y)^n$ leads to

$$2y^2\Delta\theta_n = -n\theta_{n-1} \quad .$$

Applying the L-transform to the above equation we get the differential equation

$$\Theta'(x) + \Theta(x)(2y^2+x)/x^3 = 2y^2/x^3 \quad ,$$

where

$$\Theta(x) = L[\theta_n;x] \ .$$

The solution of the above differential equation is

$$x\Theta(x)e^{-y^2/x^2} = 2y^2 \int_o^x e^{-y^2/u^2} \frac{du}{u^2} \quad .$$

If we perform the change of variables $y^2/u^2 = y^2/x^2 + t$ we will obtain

$$x\Theta(x) = y \int_o^\infty e^{-t}(t + y^2/x^2)^{-(1/2)}dt \quad ,$$

and by repeated integration by parts, we obtain

$$\Theta(x) = {}_2F_o(\tfrac{1}{2}, 1; -; -x^2/y^2) \quad ,$$

so that

$$\theta_n(y) = \sum_{j=o}^\infty (\tfrac{1}{2})_j \binom{n}{2j}(-y^{-2})^j \quad ,$$

and

$$H_n(y) = (2y)^n \ {}_2F_o(-\tfrac{n}{2}, -\tfrac{n}{2} + \tfrac{1}{2}; -; -y^{-2}) \ ,$$

which is a standard hypergeometric form of $H_n(y)$.

Our second example is to solve the three term recurrence relation

(2.40) $$(n+1)y_{n+1}(u) = (2n+1-u+\alpha)y_n(u)-(n+\alpha)y_{n-1}(u), \quad y_o = 1, \ y_{-1} = 0,$$

whose solution is known to be the Laguere polynomial $L_n^{(\alpha)}(u)$. The recurrence relation (2.40) may be written as

$$\beta y_n(u) = -uy_n + \alpha\nabla y_n \quad ,$$

whose L transform is

$$Y'(x) = -uY(x) + \alpha Y(x)/(x+1) \quad .$$

Therefore

$$Y(x) = (1+x)^{\alpha}\, e^{-xu} \quad ,$$

which is the standard generating function

$$\sum L_n^{(\alpha)}(x)t^n = (1-t)^{-\alpha-1}\, e^{-xt/1-t} \quad .$$

As another example we solve the three term recurrence relation

$$(2.41) \qquad (n+1)y_{n+1}(u) = (2n+1)uy_n(u)-ny_{n-1}(u),$$
$$y_{-1}(u) = 0 \quad \text{and} \quad y_o(u) = 1 \quad .$$

The solution is known to be the n^{th} Legendre's polynomial $P_n(u)$. Rewrite (2.41) in the form

$$\beta y_n = (2n+1)(u-1)y_n \quad ,$$

and apply the L-transform to get

$$Y'(x) = \frac{(1+2x)(u-1)}{1+(1-u)(2x+2x^2)}\, Y(x) \quad .$$

Therefore

$$Y(x) = \{1+(1-u)(2x+2x^2)\}^{-(1/2)} \quad , \tag{2.42}$$

which is nothing but the generating function

$$\sum_{o}^{\infty} P_n(x)t^n = (1-2xt+t^2)^{-(1/2)} \quad .$$

Furthermore, by expanding the right hand side of (2.42) in powers of x we obtain

$$Y(x) = \sum_{k=o}^{\infty} \frac{(\frac{1}{2})_k}{k!} (u-1)^k (2x)^k \sum_{j=o}^{k} \binom{k}{j} x^j \quad ,$$

and by applying the inverse transform L^{-1} we get

$$P_n(x) = \sum_{k,j} \binom{k}{j} \frac{(\frac{1}{2})_k}{k!} (x-1)^k \, 2^k \binom{n}{k+j} \quad ,$$

which is equivalent to

$$P_n(x) = \sum_{k=o}^{n} \frac{(\frac{1}{2})_k \, 2^k (1-x)^k \Gamma(n+k+1)}{\Gamma(2k+1} \binom{n}{k} = {}_2F_1(-n,n+1;1;\frac{1-x}{2}) \ .$$

One can derive another hypergeometric form for the Legendre polynomials by using the substitution $y_n(u) = u^n f_n(u)$. The f's will then satisfy

$$(n+1)f_{n+1}(u)-(2n+1)f_n(u) + \frac{n}{u^2} f_{n-1}(u), \quad f_o = 1,\ f_{-1} = 0 \quad ,$$

whose L transform is

$$F'(x) = x(1-u^{-2})F(x)/\{1-x^2(1-u^{-2})\} \ .$$

Thus

$$F(x) = \{1-x^2(u^2-1)/u^2\}^{-\frac{1}{2}} = \sum_{k=o}^{\infty} \frac{(\frac{1}{2})_k}{k!} \left(\frac{u^2-1}{u^2}\right)^k x^{2k} \quad ,$$

and we obtain

$$P_n(u) = u^n \sum_{k=o}^{[n/2]} \frac{(\frac{1}{2})_k}{k!} \binom{n}{2k} \left(\frac{u^2-1}{u^2}\right)^k = u^n \; {}_2F_1\left(-\frac{n}{2}, -\frac{n}{2}+\frac{1}{2}; 1; \frac{u^2-1}{u^2}\right)$$

Our next application is to solve the non homogeneous linear difference equation with constant coefficients using the L transform. The usual method of solving this type of difference equation relies on solving the characteristic equation as well as obtaining a particular solution. However, solving an algebraic equation or expressing its roots in terms of the coefficints is indeed very difficult in general. Our approach avoids this difficulty and we obtain the general solution expressed in terms of the coefficients in the difference equation.

Let

$$(2.43) \qquad \sum_{i=o}^{k} (-1)^{k-i} a_{k-i} \Delta^i f(n) = g(n) \quad , \quad k > 0 \; .$$

We may assume $a_o = 1$. The image of (2.43) under L is

$$(-1)^k a_k F(x) + \sum_{1}^{k} (-1)^{k-i} a_{k-i} \{F(x)-F(0)-\ldots-x^{i-1}F^{(i-1)}(0)\}x^{-i} = G(x) \quad ,$$

where, as usual $F(x) = L[f;x]$, $G(x) = L[g;x]$. Therefore

$$(2.44) \qquad F(x) = \frac{\{x^k G(x) + \sum_{o}^{k-1} (-1)^i a_i \sum_{j=o}^{k-i-1} F^{(i)}(0) x^{i+j}\}}{\{\sum_{o}^{k} (-1)^i a_i x^i\}}$$

Let

$$(2.45) \qquad \{1 + \sum_{1}^{k} (-1)^i a_i x^i\}^{-1} = \sum_{o}^{\infty} h_j x^j \quad .$$

From (2.44) we get

$$F(x) = \{x^k G(x) + \sum_{i=o}^{k-1} (-1)^i a_i \sum_{\ell=o}^{k-i-1} x^{i+\ell} F^{(\ell)}(0)\} \{\sum_{o}^{\infty} h_j x^j\} ,$$

or

$$(2.46) \qquad f(n) = \sum_j h_j \binom{n}{j+k} * g(n) + \sum_{j=o}^{\infty} \sum_{i=o}^{k-1} \{ \sum_{\ell=o}^{k-i-1} (-1)^i a_i F^{(\ell)}(0)\} \times h_j \binom{n}{i+j+\ell} \quad .$$

It is clear that the above solution contains k arbitrary constants, $F(0), F'(0),\ldots,F^{(k-1)}(0)$, which is the right number of constants. The first term in the right hand side of (2.46) is a particular solution while the second term is general solution of the homogeneous equation. Note that the h's in (2.45) may be given explicitly as

$$h_j = \sum_{\lambda_i} (-1)^{j-\Sigma\lambda_i} \frac{(\Sigma\lambda_i)!}{\lambda_1! \ldots \lambda_k!} a_1^{\lambda_1} \ldots a_k^{\lambda_k} \quad ,$$

where the sum is taken over $\lambda_1,\ldots,\lambda_k$ with $\Sigma i\lambda_i = j$. This follows from the multinomial theorem.

As an example, consider the difference equation

$$(2.47) \qquad C_{n+2} = C_{n+1} + C_n + n^r \quad ,$$

that was studied by Weinshenk and Hoggatt [37]. They solved it by

an operator method as well as polynomial expansions but both solutions are rather lengthy. We first write (2.47) in the form

$$\Delta^2 c_n + \Delta c_n - c_n = n^r ,$$

so that $a_1 = a_2 = -1$. It is well now that

$$\sum_{0}^{\infty} F_n x^n = (1+x-x^2)^{-1} ,$$

where $F_0, F_1, \ldots$ are the Fibonacci Numbers. Thus the h_j's of (2.45) are given by $h_j = (-1)^j F_j$,

$$h_j = (-1)^j F_j \quad ,$$

and the solution (2.47) is

$$f(n) = \sum_j (-1)^j F_j \binom{n}{j+2} * n^r + AF_n + BF_{n+1} \quad ,$$

since F_n and F_{n+1} are linearly independent solutions of the corresponding homogeneous equation.

Our final application of the L transform, in this chapter, is to Bateman Polynomials $F_k(z)$. Bateman [4], [5] studied the polynomials

$$F_k(z) = {}_3F_2(-k,k+1,(z+1)/2;1,1;1) ,$$

quite extensively. In what follows we use the L transform to derive some of his results as consequences of well known properties of the Legendre polynomials $P_k(x)$. Recall

$$P_k(x) = {}_2F_1(-k,k+1;1;\frac{(1-x)}{2}) \quad .$$

Let $n = -(1+z)/2$ in $F_k(z)$ and denote the resulting function by $f_k(n)$. Clearly

(2.48) $$L[f_k(n);x] = {}_2F_1(-k,k+1;1;-x) = P_k(2x+1) \quad .$$

Relation (2.48) enables us to derive several properties of $f_k(n)$. Under L^{-1}, Legendre's differential equation

$$x(x+1)\frac{d^2}{dx^2}P_k(2x+1)+(1+2x)\frac{d}{dx}P_k(2x+1) = k(k+1)P_k(2x+1) \quad ,$$

is transformed to

$$(\binom{n}{2}+n)*\beta_n^2 f_k(n) + (1+2n)*\beta_n f_k(n) = k(k+1)f_k(n) \quad .$$

which leads to, after some manipulations,

(2.49) $$\frac{1}{4}(z-1)^2F_k(z-2) + \frac{1}{4}(z+1)^2F_k(z+2) - \frac{1}{4}\{(z-1)^2+(z+1)^2\}F_k(z) = k(k+1)F_k(z) \quad .$$

So, (2.49) is valid for negative odd integeral values of z, hence for all z.

Using this method we can also derive the generating function

$$\sum F_n(z)t^n = (1-t)^{-1}\,{}_2F_1(\tfrac{1}{2}, \tfrac{1}{2}+\tfrac{1}{2}z;1;\frac{-4t}{(1-t)^2}) \quad ,$$

from Bateman's $\{F_n(z)\}_o^\infty$ for the standard generating function

$$\sum P_n(x)t^n = (1-2xt+t^2)^{-(1/2)} \quad ,$$

for the Legendre polynomials.

The general Lucas polynomials $U_n^{(N)}(a_1,\ldots,a_N)$, or simply $U_n^{(N)}$ are defined by

$$U_{n+N}^{(N)} = a_1 U_{n+N-1}^{(N)} - a_2 U_{n+N-2}^{(N)} + \ldots - (-1)^n a_n U_n^{(N)} ,$$

with $U_0^{(N)} = U_1^{(N)} = \ldots = U_{N-2}^{(N)} = 0$, $U_{N-1}^{N} = 1$. They arose in connection with finding the M^{th} power of an $N \times N$ matrix [3]. Barakat and Baumann [3] expressed a need for a closed form which was obtained in [30]. The required closed form obviously follows from (2.46).

CHAPTER III

EXTENSIONS

We note that the finite difference analogue of the Carson-Laplace transform, that was studied in Chapter II, is a sequence to function transform known in summability as Abel transform. Since this is only a special case of a much more general sequence to function transform, the so called $[J,\phi(x)]$ transform introduced in [19], then it is of interest to see which, if any, of these general transforms yield an operational calculus, or a transform theory, along the same lines we developed in Chapter II. We shall restrict ourselves to such $\phi(x)$'s that are power series in x. To be exact let

$$(3.1) \qquad L[f;\alpha,x] = \sum_{0}^{\infty} \frac{(-x)^n}{n!} f(n) \frac{d^n}{dx^n} \phi(x) \quad ,$$

where

$$(3.2) \qquad \phi(x) = \sum_{0}^{\infty} \alpha_n x^n \quad .$$

Substituting for $\phi(x)$ in (3.1) and using

$$(3.3) \qquad f(n) = \sum_{0}^{n} \binom{n}{k} \Delta^k f(0) \quad ,$$

we get

$$(3.4) \qquad L[f;\alpha,x] = \sum_{n=0}^{\infty} (-x)^n \alpha_n \Delta^n f(0) \quad ,$$

provided that we can rearrange the terms in the series appearing in (3.1). At any rate for a given sequence $\alpha : \alpha_0, \alpha_1, \ldots, \alpha_n, \ldots$ we shall take (3.4) as the definition of our transform instead of (3.1). The dependence on the sequence $\{\alpha_n\}_o^\infty$ is explicit in the notation $L[f;\alpha,x]$. If $\phi(x)$ has a positive radius of convergence then (3.1) and (3.4) will be equivalent.

In the case $\phi(x) = (1+x)^{-1}$, or rather $\alpha_n = (-1)^n$, we have the L transform of Chapter II. Whenever we drop the dependence on $\{\alpha_n\}_o^\infty$, that is we write $L[f;x]$, we shall always mean $L[f;(-1)^n,x]$ of Chapter II.

We note that formally (3.1) is equivalent to

$$L[f;\alpha,x] = \sum_{n=o}^{\infty} (-x)^n f(x) \sum_{j=o}^{\infty} \binom{n+j}{j} \alpha_{n+j} x^j \quad ,$$

which in turn has the operational representation

$$L[f;\alpha,x] = \sum_{n=o}^{\infty} f(n) \frac{(-xE)^n}{(1+xE)^{n+1}} \alpha_o \quad ,$$

from which we see that

$$L[f;\alpha,x] = F(-xE)\ \alpha_o$$

where $F(x) = L(f;x)$.

As before, we shall use small letters $f, g, \ldots$ to denote sequences and use capital letters $F, G, \ldots$ to denote their transforms. We shall refer to these transforms as (L,α) transforms in general or (L,α_n) if we are spedifying the sequence $\{\alpha_n\}_o^\infty$.

It is natural to require (L,α) to satisfy

$$L[1;\alpha,x] = 1 \ , \quad \text{where} \quad \{1\} = \{1,1,\ldots\} \ ,$$

which is the case if and only if $\alpha_o = 1$. We also require (L,α) to be one-to-one. Relation (3.3) shows that f is uniquely determined by $f(0), \Delta f(0), \ldots,$ hence our transform is one-to-one if and only if none of the α_n's vanish. That is why we shall assume

$$\alpha_o = 1 \quad \text{and} \quad \alpha_n \neq 0 \quad n = 1,2,\ldots \ . \tag{3.5}$$

Now we look for a suitable convolution product and a finite difference operator β_α such that

$$L[\beta_\alpha f;\alpha,x] = \frac{d}{dx} L[f;\alpha,x] \ , \tag{3.6}$$

and

$$\beta_\alpha(f*g) = f*(\beta_\alpha g) + (\beta_\alpha f)*g \ . \tag{3.7}$$

Recall that we referred, and shall always refer, to the case where $\alpha_n = (-1)^n$ as β.

3.1 Properties of the (L,α) Transform. We define the sum of two sequences as before. The requirement

$$L[f*g;\alpha,x] = F(x)\ G(x) \ , \tag{3.8}$$

defines the convolution product uniquely. In fact we have

Theorem 3.1 *The relation (3.8) is satisfied for all* $f,g \in S$ *if and only if*

$$(3.9) \qquad f*g(n) = \sum_{r=0}^{n} \binom{n}{r} \sum_{i=0}^{r} \frac{\alpha_i \alpha_{r-i}}{\alpha_r} \Delta^i f(0) \Delta^{r-i} g(0) \quad .$$

Proof. If (3.8) is satisfied, then

$$\sum_{0}^{\infty} (-x)^n \alpha_n \Delta^n (f*g)(0) = \{ \sum_{i=0}^{\infty} \alpha_i (-x)^i \Delta^i f(0) \} \{ \sum_{j=0}^{\infty} \alpha_j (-x)^j \Delta^j g(0) \} ,$$

so that

$$\alpha_n \Delta^n f*g(0) = \sum_{i=0}^{n} \alpha_i \alpha_{n-i} \Delta^i f(0) \Delta^{n-i} g(0) \quad ,$$

and (3.9) follows by (3.3)

Conversely, if (3.9) is satisfied, then (3.8) will follow by elementary series manipulations.

So, we adopt (3.9) as the definition of the convolution product. Let us define a sequence $c_o, c_1, \ldots$ by

$$(3.10) \qquad c_o = 0 \quad \text{and} \quad c_n = -\alpha_n / \alpha_{n-1} , \qquad n = 1,2,\ldots \quad ,$$

so that

$$(3.11) \qquad \alpha_n = (-1)^n c_1 \cdots c_n , \quad n > 0 \quad \text{and} \quad \alpha_o = 1 ,$$

and introduce two linear operators A_α and β_α by

$$(3.12) \qquad A_\alpha f(n) = \sum_{k=1}^{n+1} c_k \binom{n}{k-1} \Delta^k f(0) ,$$

and

$$(3.13) \qquad \beta_\alpha = A_\alpha n \nabla , \quad \text{i.e,} \ (\beta_\alpha f)(n) = A_\alpha n \nabla f(n) .$$

The operator β_α satisfies

$$\beta_\alpha f(n) = \sum_{k=o}^{n} (k+1) c_{k+1} \binom{n}{k} \Delta^{k+1} f(0) \quad . \tag{3.14}$$

To see this, we have

$$\beta_\alpha f(n) = A[n\nabla f(n)] = \sum_{k=o}^{n} \binom{n}{k} c_{k+1} \Delta^{k+1} [m(f(m)-f(m-1))]_{m=o} \quad ,$$

and by the finite difference analogue of Leibnitz formula, see ([34], p. 11), we obtain

$$\beta_\alpha f(n) = \sum_{k=o}^{n} \binom{n}{k} c_{k+1} \{m\Delta^{k+1}(f(m)-f(m-1))+(k+1)\Delta^{k}(f(m+1)-f(m))\}_{m=o}$$

$$= \sum_{k=o}^{n} \binom{n}{k} (k+1) c_{k+1} \Delta^{k+1} f(0) \quad ,$$

proving (3.14).

It is clear, from (3.12) and (3.14), that for the sequence $\{\binom{n}{k},\ n = 0,1,2,3,\ldots\}$ we have

$$A_\alpha \binom{n}{k} = c_k \binom{n}{k-1} \quad , \tag{3.15}$$

and

$$\beta_\alpha \binom{n}{k} = k\, c_k \binom{n}{k-1} \quad , \tag{3.16}$$

It will turn out the β_α as defined by (3.13) is the sought operator of (3.6) and (3.7)

<u>Theorem 3.2.</u> <u>The following relations</u>

$$L[\binom{n}{r} \frac{(-1)^r}{\alpha_r};\alpha,x] = x^r \quad , \tag{3.17}$$

$$(3.18) \qquad L[A_\alpha f;\alpha,x] = \frac{F(x)-F(0)}{x} ,$$

as well as (3.6) hold.

Proof. (3.17) follows from $\Delta^m \binom{n}{r}\Big|_{n=o} = \delta_{r,m}$ and (3.4). By (3.13) and (3.3) we get

$$\Delta^m A_\alpha f(0) = c_{m+1}\Delta^{m+1} f(0) = -\frac{\alpha_{m+1}}{\alpha_m}\Delta^{m+1} f(0) \quad .$$

Thus

$$L[A_\alpha f;\alpha,x] = \sum_{m=o}^{\infty} (-1)^{m+1}\alpha_{m+1} x^m \Delta^{m+1} f(0) = \frac{F(x)-F(0)}{x} \quad ,$$

proving (3.18). Similarly (3.6) follows from (3.14) and (3.3).

Corollary. β_α satisfies (3.7).

Proof. This follows by (3.6) and (3.8). It can be proved directly from (3.9) and (3.14).

Note that if c_n is a polynomial in n, then A_α will be a finite difference operator, for example $c_n = 1, n, n+\gamma$ correspond to $A_\alpha = \Delta, \Delta n\nabla$ and $\gamma\Delta + \Delta n\nabla$ respectively. In fact we have

Theorem 3.3. If $c_n = P(n)$, where P is a polynomial, then $A_\alpha = P(\tau)\Delta$ and $B_\alpha = P(\tau)\beta$, where $\tau = 1+x\nabla$.

The proof is rather easy and is omitted. These operational formulas can be extended formally to functions which are power series.

At this stage we can use the (L,α) transform to solve equations in A_α of the type

$$\sum_{i=o}^{r} b_i A_\alpha^i f(n) = g(n) \quad ,$$

with constant coefficients $b_0, b_1, \ldots, b_r$. Let us illustrate the method and its advantages by an example.

<u>Example.</u> Consider the equation

$$\Delta n \nabla f(n) - f(n) = g(n) \quad . \tag{3.19}$$

We take $A_\alpha = \Delta n \nabla$, that is $c_n = n$, so that

$$F(x) = L[f;(-1)^n\, n!,x] = \sum_{n=o}^{\infty} n!\; x^n \Delta^n f(0) \quad .$$

Applying the $(L,(-1)^n(n!))$ transform to (3.19) and using (3.18) we get

$$F(x) = \frac{f(0)}{1-x} + \frac{xG(x)}{1-x} \;, \quad \text{with} \quad G(x) = L[g;(-1)^n\, n!,x] \quad .$$

Thus, using (3.17) we obtain

$$f(n) = f(0) \sum_{k=o}^{n} \binom{n}{k}/k! + \sum_{k=1}^{n} \binom{n}{k} * g(n)/k! \quad .$$

The observation

$$\Delta^i \sum_{k=o}^{n} \binom{n}{k}/k! \Big|_{n=o} = (1-\delta_{i,o})/i! \quad ,$$

leads to

$$f(n) = f(0) \sum_{k=o}^{n} \binom{n}{k}/k! + \sum_{k=1}^{n} \binom{n}{k} \sum_{i=o}^{k-1} i!\,\Delta^i g(0)/k! \quad .$$

On the other hand if we use a generating function method, i.e., if

we put $\theta(t) = \sum_{0}^{\infty} f(n)t^n$, to solve (3.19), we will get

$$\theta(t) = \frac{e^{1/(1-t)}}{(1-t)} \int_0^t e^{-1(1-u)} \eta(u) \frac{du}{1-u} ,$$

where $\eta(t) = \sum_{0}^{\infty} g(n)t^n$. Now it is apparent that recovering $f(n)$ from $\theta(t)$ involves very messy expansions. The use of an exponential generating function leads to a second order differential equation. The use of the L transform of Chapter II leads to

$$L[f;x] = e^x \int_0^x e^{-t} L[g;t]dt ,$$

and recovering $f(n)$ will again involve messy expansions.

Now we go back to the general (L,α) transforms and look at the transform of $nf(n)$ and $\Delta f(n)$. First, we introduce the linear operators C and $\mathcal{C}$, defined on formal power series by

(3.20) $$C\, x^n = x^{n-1}/c_n \quad \text{if} \quad n > 0 , \quad C \cdot 1 = 0$$

(3.21) $$\mathcal{C}\, x^n = (n+1)c_{n+1}\, x^n , \qquad n = 0,1,\ldots .$$

<u>Theorem 3.4.</u> <u>Under general</u> (L,α) <u>transforms we have</u>

(3.22) $$L[\Delta f;\alpha,x] = C(L[f;\alpha,x]) ,$$

<u>and</u>

(3.23) $$L[nf(n);\alpha,x] = x\frac{d}{dx}L[f;\alpha,x] + x\mathcal{C}(L[f;\alpha,x]) .$$

<u>Proof.</u> We have

$$L[\Delta f;\alpha,x] = \sum_{o}^{\infty} (-x)^n \alpha_n \Delta^{n+1} f(0) = \sum_{o}^{\infty} (-1)^n \alpha_n C[x^{n+1}] \cdot c_{n+1} \Delta^{n+1} f(0)$$

$$= \sum_{1}^{\infty} (-1)^{n+1} \alpha_{n+1} \Delta^{n+1} f(0) C[x^{n+1}] = C(L[f;\alpha,x]) ,$$

proving (3.22). Now

$$\Delta^m \, nf(n)\Big|_{n=o} = m\,\Delta^{m-1} f(1) ,$$

follows from the finite difference analogue of Leibnitz formula ([34], p. 11). Therefore, since $m\,\Delta^{m-1} f(1) = m\Delta^m f(0) + m\Delta^{m-1} f(0)$,

$$L[nf(n);\alpha,x] = \sum_{o}^{\infty} (-x)^n \alpha_n \{n\Delta^n f(0) + n\Delta^{n-1} f(0)\}$$

$$= x \frac{d}{dx} \{ \sum_{o}^{\infty} (-x)^n \alpha_n \Delta^n f(0) \}$$

$$+ \sum_{o}^{\infty} (-x)^{n+1} \alpha_n (-C_{n+1})(n+1) \Delta^n f(0)$$

= the right hand side of (3.23).

If $\sup \{|\alpha_n \Delta^n f(0)|^{1/n} : n = 1,2,\ldots\} < \infty$, the function $F(x)$ will be holomorphic in some neighborhood of the origin. Let

$$Y = \{\{f_n\}_o^\infty : \sup_{m>o} |\alpha_m \Delta^m f(0)|^{1/m} < \infty\} .$$

Y is closed under both addition and the convolution product (3.9). For, by (3.9) we get for $m > 0$

$$|\alpha_m \Delta^m f*g(0)|^{1/m} \le (m+1)^{1/m} \max \{|\alpha_i \Delta^i f(0)|^{1/m} : 0 \le i \le m\}$$

$$\times \max \{|\alpha_j \Delta^j g(0)|^{1/m} : 0 \le j \le m\}$$

$$\le 2\nu^2 ,$$

where $\nu = \max \{1, f(0), g(0), \sup_{i>o} |\alpha_i \Delta^i f(0)|^{1/i}, \sup_{j>o} |\alpha_j \Delta^j f(0)|^{1/j}\}$. Thus $f*g \in Y$ for $f \in Y$ and $g \in Y$.

By using the argument in Theorem 3.1, we can prove (3.8), for $f \in Y$, where, in this setting, equalities as (3.6), (3.8), ... etc. must be interpretted as equalities between holomorphic functions. With this in mind, Theorem 3.2 is valid if the left hand side of (3.6), (3.17) and (3.18) are holomorphic in a neighborhood of the origin. This is indeed so since $\Delta^m \binom{n}{r}\Big|_{n=o} = \delta_{r,m}$, $\alpha_m \Delta^m A_\alpha f(0) = -\alpha_{m+1} \Delta^{m+1} f(0)$, and $\alpha_m \Delta^m \beta_\alpha f(0) = -(m+1)\alpha_{m+1} \Delta^{m+1} f(0)$ imply $\{\binom{n}{r}, r = 0,1,\ldots\} \in Y$ and that both $A_\alpha f$ and $\beta_\alpha f$ are members of Y whenever f is so.

This approach is undoubtedly more restrictive than our former one, using formal power series. On the other hand the present approach allows us to use complex function theory. For example the inversion formula

$$\Delta^n f(0) = \frac{F(z)}{2\pi i \alpha_n} \int_i \frac{F(z)}{z^{n+1}}\, dz \quad , \tag{3.24}$$

or equivalently

$$(3.25) \qquad f(n) = \frac{1}{2\pi i} \int_c \{ \sum_{k=o}^{n} (-1)^k \binom{n}{k} z^{-k-1}/\alpha_k \} F(x)dz ,$$

is obvious, with $F(z)$ holomorphic in and on the contour c.

If we restrict ourselves to operators A_α, or sequences $\{\alpha_n\}_o^\infty$, for which $0 < \sup_{n>o} |c_n|^{1/n} < \infty$, then both $CF(x)$ and $CF(x)$ are well defined and holomorphic in a neighborhood of the origin for $f \in Y$. Moreover, Y will be closed under finite differences and products of its members by polynomials. In this case (3.22) and (3.23) follow by a legitimate rearrangement of terms in the Taylor series of the respective right hand sides.

3.2. An Operational Calculus Approach. In this section we develop an operational calculus based on the convolution product (3.9). Our approach is, as before, an algebraic approach, see [16]. It is easy to see that the set S under addition and the convolution product (3.9) forms an integral domain. We shall denote its field of quotients by Q.

Theorem 3.5. We have

$$(3.26) \qquad A_\alpha(\{\frac{n}{c_1}\}*f) = f ,$$

and

$$(3.27) \qquad \{\frac{n}{c_1}\}*A_\alpha f = f \qquad \text{whenever} \qquad f(0) = 0 .$$

Proof. It is easy to see that $\Delta^i f(0) = \delta_{i,1}$ if $f(n) = n$. Thus

$$A_\alpha(\{\frac{n}{c_1}\}*f) = A\,\{\sum_{r=1}^{n} \binom{n}{r} \frac{\alpha_{r-1}\alpha_1}{c_1\,\alpha_r} \Delta^{r-1} f(0)\} = f \quad .$$

Relation (3.27) can be proved similarly.

Relation (3.27) shows that A_α can be identified with the quotient $\{1\}/\{n\}$ on sequences $\{f(n)\}_o^\infty$ with $f(0) = 0$.

Definition. The index of a sequence s_n, denoted by $i(s_n)$ or $i(s)$, the index of its first non-zero term, that is $i(s_n) = \min\{n : s_n \neq 0\}$.

It is clear that $i(a_n) = \min\{i : \Delta^i a_o \neq 0\}$. The useful relation

$$i(a*b) = i(a) + i(b) \quad , \tag{3.28}$$

follows from

$$\alpha_r \Delta^r f*g(0) = \sum_{o}^{r} \alpha_i \alpha_{r-i} \Delta^i f(0) \Delta^{r-i} g(0) \quad ,$$

which in turn follows from (3.9).

Theorem 3.6. In order that a convolution quotient $\frac{a}{b}$ belong to S, it is necessary and sufficient that $i(a) \geq i(b)$.

Proof. If $c = \frac{a}{b} \in S$, then $b*c = a$ and so $i(b)+i(c) = i(a)$, and the necessity follows.

Conversely, if $i(a) \geq i(b)$, then the system of equations

$$\alpha_r \Delta^r a_o = \sum \alpha_i \alpha_{r-i} \Delta^i x_o \Delta^{r-i} b_o \ , \quad r = 0,1,2,\ldots$$

in the unknowns x_o, Δx_o, ..., $\Delta^n x_o$, ... has a unique solution, proving the sufficiency.

As before, we denote $\underbrace{f*f*\ldots*f}_{n\text{-times}}$ by $f^{(n)}$.

Theorem 3.7. Let $\{s_n\} \in S$ with $s_n \not\equiv 1$, then $\lim_{n\to\infty} f^{(n)}$ exists if and only if $|s_o| < 1$, in which case the limit will be zero.

The proof of this result is similar to the proof of Theorem 2.3 and is omitted.

1.3 Special Cases. In this section we study two special (L,α) transforms. Recall that the transform L of Chapter II is another special case. For more applications see Chapter IV.

1.3.1 The Case. $\alpha_n = (-1)^n/n!$. In this case $c_n = 1/n$ for $n > 0$, or $\phi(x) = e^{-x}$. This special transform, $L[f;(-1)^n/n!,x]$ may be defined by

$$L[f;(-1)^n/n!,x] = e^{-x} \sum_{o}^{\infty} \frac{x^n}{n!} f(n) \quad ,$$

and is indeed a modified exponential generating function. This transform, apart from the factor e^{-x}, was studied by Berge [8]. In this case the convolution product (3.9) reduces to

$$f*g(n) = \sum_{r=o}^{n} \binom{n}{r} \Delta^r g(0) f(n-r) \quad ,$$

the operator C is the differentiation operator and $\mathcal{C}$ is the identity. Thus we have

$$(3.29) \qquad L[\Delta f;\frac{(-1)^n}{n!},x] = DL[f;\frac{(-1)^n}{n!},x] \quad ,$$

and

$$(3.30) \qquad L[nf(n);\frac{(-1)^n}{n!},x] = xL[f;\frac{(-1)^n}{n!},x] + xD[f;\frac{(-1)^n}{n!},x] \quad .$$

For more properties and numerous applications the reader is referred to [8].

As an illustration let us solve the recurrence

$$(3.31) \qquad f_{n+2} = (n+4)f_{n+1}-(n+1)f_n \quad (n > 0), \quad \text{with} \quad f_1 = a \quad \text{and} \quad f_2 = b$$

which appeared in the advanced problem section, Monthly, vol 80 (1973), problem 5911.

First we rewrite (3.31) as

$$\Delta^2 f_n = n\Delta f_n+2\Delta f_n+2f_n+(b-4a)\delta_{n,o}, \quad n = 0,1,\ldots \quad .$$

By (3.29) and (3.30) we obtain, for $F(x) = L[(-1)^n/n!,f,x]$, the differential equation

$$(3.32) \qquad (1-x)F''(x)-(2+x)F'(x)-2F(x) = (b-4a)e^{-x} \quad .$$

Multiplying (3.32) by e^x and integrating the result we get

$$e^x(1-x)F'(x)-2e^xF(x) = (b-4a)x + \text{constant} \quad ,$$

and the initial conditions imply

$$(3.33) \qquad (1-x)F'(x)-2F(x) = (b-4a)xe^{-x}+ae^{-x} \quad .$$

The solution of (3.33) is

$$(1-x)^2 F(x) = \int_0^x \{(b-4a)t+a\}e^{-t}(1-t)dt + \text{constant} .$$

Therefore we get

$$F(x) = \frac{e^{-x}}{(1-x)^2}\{(b-4a)x^2+(b-3a)x+(b-4a)\} - \frac{(b-4a)}{(1-x)^2} ,$$

and

$$f_n = 4a-b+n(n!)\{3b-11a+(4a-b)\sum_0^n (j!)^{-1}\} .$$

As an application of this transform we derive Kummer's transformation

$${}_1F_1(a;b;x) = e^x \, {}_1F_1(b-a;b;-x) .$$

We write

$${}_1F_1(a;b;-x) = \sum_0^\infty \frac{(a)_k(-1)^k}{(b)_k\, k!} x^k = L[\sum_{k=0}^\infty \frac{(a_k)(-n)_k}{(b)_k\, k!}; \frac{(-1)^n}{n!}, n]$$

$$= L[\frac{(b)_n}{(b-a)_n}; \frac{(-1)^n}{n!} ;x] = e^{-x}\sum_0^\infty \frac{(b)_n}{n!\,(b-a)_n} x^n$$

$$= e^{-x} \, {}_1F_1(b-a,b,x) .$$

1.3.2. <u>The Special Case</u>. $\alpha_n = (-1)^n\binom{\gamma+n}{n}$. In this case $c_n = \frac{\alpha+n}{n}$, $n > 0$. We also assume that α is not a negative integer. We shall denote the operators A_α and β_α in this case by A_γ and β_γ and shall not use the symbol γ anywhere else. Clearly

(3.34) $$A_\gamma f(n) = \Delta f(n) + \gamma \frac{f(n+1)-f(0)}{n+1}$$

and

(3.35) $$\beta_\gamma f(n) = \Delta(\gamma+n\nabla)f(n) \quad .$$

It is clear that our β_γ is the finite difference analogue of the generalized Bessel operator $t^{-\gamma}\frac{d}{dt}t^{\gamma+1}\frac{d}{dt}$ introduced by Meller [21], since the latter can be expressed as $\frac{d}{dt}(\gamma + t\frac{d}{dt})$. Note that in case $\gamma = 0$, the operator A_γ reduces to Δ and hence our operational calculus and transforms reduce to those of Chapter II.

For simplicity we use $J_\gamma[f;x]$ to denote $L[f;(-1)^n\binom{\gamma+n}{n},x]$. Clearly

(3.36) $$J_\gamma[f;x] = \sum_0^\infty x^n\binom{\gamma+n}{n}\Delta^n f(0) \quad ,$$

and

(3.37) $$J_\gamma[f;x] = \sum_0^\infty \frac{x^n\binom{\gamma+n}{n}}{(1+x)^{\gamma+n+1}} f(n) \quad .$$

Relation (3.37) shows that the J_γ transform corresponds to the A_λ method of summation, a generalization of the Abel means, introduced in [9]. By straightforward manipulations, we obtain

(3.38) $$J_\gamma[\nabla f(n) + \frac{\gamma}{n+\gamma} f(n-1);x] = J_\gamma[f;x]/(x+1) \quad .$$

Relation (3.38) is a characteristic property of the J_γ transform. Indeed we have

Theorem 3.8. *If an* (L,α) *transform satisfies*

$$L[\nabla f(x) + \frac{\gamma}{n+\gamma} f(n-1);\alpha_n,x] = h(x)L[f;\alpha,x] \quad ,$$

whenever $L[f;\alpha,x]$ *and* $h(x)$ *are holomorphic in a neighborhood of the origin and* γ *is not a negative integer, then* $h(x) = (1+x)^{-1}$ *and* (L,α) *is* J_γ, *up to a change of variable* $x \to \lambda x$.

Proof. It is clear that $L[f;\alpha,x]$ will be holomorphic in a neighborhood of the origin if $f(n)$ vanishes eventually. For those sequences we have, by (3.3)

$$L[\nabla f(n)+\frac{\gamma}{n+\gamma} f(n-1);\alpha_n,x] = \sum_{n,k=o}^{\infty} (-1)^n x^{n+k}\alpha_{n+k}\{f(n) - \frac{n}{n+\gamma}f(n-1)\}\binom{k+n}{n}$$

$$= \sum_{n,k=o}^{\infty} (-1)^n x^{n+k}\binom{n+k}{n}f(n)\{\alpha_{n+k} + \frac{n+k+1}{n+\gamma+1}\, x\alpha_{n+k+1}\} .$$

On the other hand

$$L[\nabla f(n) + \frac{\gamma}{n+\gamma} f(n-1);\alpha_n,x] =$$

$$= \{\sum_{o}^{\infty} h_k x^k\}\{\sum_{n,\ell=o}^{\infty} x^{n+\ell}(-1)^n\binom{\ell+n}{n}f(n)\alpha_{n+\ell}\},$$

by assumption, where $h(x) = \sum_{o}^{\infty} h_k x^k$. Therefore by taking $f(k) = \delta_{k,j}$, we obtain

$$\sum_{k=o}^{\infty} x^k\binom{j+k}{j}\{\alpha_{k+j} + \frac{j+k+1}{j+\gamma+1}\, x\alpha_{j+k+1}\}$$

$$= \{\sum_{j}^{\infty} h_k x^k\}\{\sum_{\ell=o}^{\infty} x^\ell\binom{\ell+j}{j}\, \alpha_{j+\ell}\} .$$

Equating coefficients of powers of x on both sides of the above

identity we get

$$h_o = 1 \quad \text{and} \quad \alpha_{n+1} = \left(\frac{n+\gamma+1}{n+1}\right) h_1 \alpha_n \quad .$$

Therefore

$$c_{n+1} = -h_1 \left(\frac{n+\gamma+1}{n+1}\right) \quad ,$$

hence $h_1 \neq 0$ and we may assume $h_1 = -1$ since we allow the change of variable $x \to \lambda x$. Consequently the (L,α) transform is nothing but the J_γ transform. That $h(x) = (1+x)^{-1}$ follows from (3.38).

Corollary. The only (L,α) transform that satisfies

$$L[\nabla f;\alpha,x] = h(x)L[f;\alpha,x] \quad ,$$

where $L[f;\alpha,x]$ and $h(x)$ are holomorphic in a neighborhood of the origin is the L transform of Chapter II.

As an application of this special transform, we use it to derive the formula

$$(3.39) \qquad {}_2F_1(a,b;c;z) = (1-z)^{-a} \; {}_2F_1\left(a,c-b;c; -\frac{z}{1-z}\right) \quad .$$

Let f be defined by

$$(3.40) \qquad J_\gamma[f;x] = \sum_{o}^{\infty} \frac{(1+\gamma)_k (b)_k}{k! \; (c)_k} (-x)^k \quad .$$

Therefore

$$f(n) = \sum_{k=o}^{\infty} \frac{(b)_k (-n)_k}{k! \; (c)_k} = (c-b)_n/(c)_n \quad ,$$

since J_γ^{-1} maps x^k to $\frac{n^{(k)}}{(1+\gamma)_k}$. Thus

$$J_\gamma[f;x] = \sum_{n=o}^{\infty} \frac{\binom{\gamma+n}{n} (c-b)_n \, x^n}{n! \, (c)_n \, (1+x)^{n+\gamma+1}} \tag{3.41}$$

$$= (1+x)^{-\gamma-1} \, {}_2F_1(\gamma+1, c-b; c; \tfrac{x}{1+x}) \quad .$$

Relation (3.39) now follows from (3.40) and (3.41).

CHAPTER IV

ORTHOGONALITY WITH RESPECT TO CONVOLUTION

In this chapter we introduce and investigate a discrete analogue of the concept of "Orthogonality with respect to convolution introduced in [2]. We also point out discrete analogues of Rodrigues' formulas for certain polynomial sets. The last section is devoted to proving, under some more assumptions that the zeros of polynomials that are orthogonal with respect to convolution are real and simple.

4.1 Convolution Orthogonality. Let $\psi(x)$ be a non-decreasing function of bounded variation on $(-\infty,\infty)$ such that all its moments $\int_{-\infty}^{\infty} t^j d\psi$, $j = 0,1,\ldots,$ exist and has infinitely many points of increase; that ψ is a distribution function; and let $\{p_j(x)\}_0^\infty$ be a polynomial set orthogonal with respect to ψ, that is

$$(4.1) \qquad \int_{-\infty}^{\infty} p_j(x)\; p_\ell(x)d\psi(x) = \lambda_j\delta_{j,\ell} \quad ,$$

and let $L[f;\alpha_n,x]$ be an (L,α) transform. If $Q_j(n)$ is the preimage under (L,α_n) of $p_j(x)$, then the orthogonality relation (4.1) is equivalent to

$$(4.2) \qquad \int_{-\infty}^{\infty} L[Q_j * Q_\ell;\alpha_n,x]d\psi(x) = \lambda_j\delta_{j,\ell} \; .$$

Relation (4.2) expresses a kind of orthogonality for the polynomials $Q_0(x),Q_1(x), \ldots$. We shall refer to this type as "Orthogonality with respect to convolution" or simply "Convolution orthogonality".

To be specific, let $\{Q_j(x)\}_o^\infty$ be a simple set of polynomials. We say that this polynomial set is orthogonal with respect to convolution if there exist a distribution function $\psi(x)$ and a convolution $*$ induced by an (L,α) transform that satisfy (4.2) for some sequence $\lambda_0,\lambda_1,\ldots$ of positive numbers. In other words a polynomial set $\{Q_j(x)\}_{j=o}^\infty$ is orthogonal with respect to convolution if and only if $\{L[Q_j;\alpha_n,x]\}_{j=o}^\infty$ is orthogonal in the classical sense. Note that one can drop the non-decreasing requirement of ψ and still talk about an analogous concept. However, we shall always assume that ψ is non-decreasing.

If we know a Rodrigues type formula for $p_j(x)$, then by virtue of the above correspondence and (3.6) we obtain a corresponding relation for $Q_j(n)$ involving β_α, of course. This will be made clearer in the examples of the next section.

4.2 Examples. As a first example we take $\{p_j(x)\}_o^\infty$ to be the Legendre polynomials of argument $1-2vx$, that is

$$p_j(x) = {}_2F_1(-j;j+1;1;vx) \quad , \tag{4.3}$$

and we take $\alpha_n = \frac{(p)_n}{n!}$. The $(L, \frac{(p)_n}{n!})$ transform maps $n^{(k)}$ to $(-1)^k(p)_k x^k$, so that

$$L^{-1}[p_j(x); \frac{(p)_n}{n!},n] = {}_3F_2(-j,j+1,-n;1,p;v) \quad , \tag{4.4}$$

p is not a negative integer.

The Rice polynomials $H_j(\xi,p,v)$ are ${}_3F_2(-j,j+1,\xi;1,p,v)$, see [31], p. 287. These polynomials were introduced and studied by Rice [32].

In this case, relation (4.2) for $v > 1$, is

$$(4.5) \qquad \sum_{m,n} H_j(-n,p,v)H_\ell(-m,p,v) \frac{(p)_m(p)_n}{m!\ \ n!} \mu_{n+m} = \delta_{j,\ell}/v(2j+1), \quad v > 1 ,$$

with

$$\mu_k = (-1)^k \int_0^{1/v} x^k(1-x)^{-k-2p}dx \quad .$$

The Rice polynomials $H_j(\xi,p,v)$ are not orthogonal in v unless $\xi = 1$ or p. This follows from §3 of [1]. In the cases $\xi = 1$ or p, they reduce to special Jacobi polynomials. They are orthogonal in ξ if and only if $v = 1$. The if part follows from §6 of [1] and the only if part follows from Theorem 4.3, to be proved later.

Thus the Rice polynomials for $v > 1$, is an example of a polynomial set that is orthogonal with respect to convolution and not orthogonal, even in the generalized sense [33].

Rodrigues' formula for the Legendre polynomials $P_n(x)$ is ([31], p. 161)

$$(4.6) \qquad P_n(x) = D^n(x^2-1)^n/(2^n\ n!) \quad .$$

The β_α operator corresponding to $\alpha_n = (p)_n/n!$ is $-\Delta n\nabla-(p-1)\Delta$. Therefore, by (4.3), (4.4) and (4.6) we get

$$(4.7) \qquad H_j(\xi,p,v) = \frac{j!}{(p)_j} \{\Delta\xi\nabla+(p-1)\Delta\}^j \binom{\xi}{j} {}_2F_1(-j,j+\xi;p+j;v) \quad .$$

In particular

$$(4.8) \qquad H_j(\xi,p,1) = \frac{j!}{(p)_{2j}} \{\Delta\xi\nabla+(p-1)\Delta\}^j \binom{\xi}{j}\binom{\xi-p}{j} \quad .$$

The Bateman's polynomials $F_n(z)$ are $H_n(z,1,1)$. In this case (4.8) reduces to the rather simple form

$$F_n(z) = \frac{j!}{(2j)!} (\Delta z \nabla)^j \binom{z}{j}\binom{z-1}{j} \quad . \tag{4.9}$$

Our second example is the Meixner polynomials $m_j(x; \ ,c)$ defined by ([15], vol. 2, p. 225).

$$m_j(x;\beta,c) = (\beta)_j \ {}_2F_1(-j,-x;\beta;1-c^{-1}) \quad . \tag{4.10}$$

They are related to the Laguerre polynomial $L_j^{(\beta)}(x)$ as

$$L[m_j(n;\beta+1,b+1);(-1)^n/n!,x] = j! \ L_j^{(\beta)}(bx) \quad , \tag{4.11}$$

since

$$L_j^{(\beta)}(x) = \frac{(\beta+1)_j}{j!} \ {}_1F_1(-j;\beta+1);x) \quad . \tag{4.12}$$

The orthogonality relation (4.1) for the Laguerre polynomials is

$$\int_0^\infty e^{-bx} x^\beta L_j^{(\beta)}(bx) L_\ell^{(\beta)}(bx)dx = b^{\beta+1}\Gamma(\beta+j+1)\delta_{j,\ell}/j! \quad , \tag{4.13}$$

$$\operatorname{Re} \beta > -1 \ .$$

Thus, for $\operatorname{Re} \beta > -1$, we have

$$\sum_{r,n=0}^{\infty} m_j(n;\beta+1,(b+1)^{-1}) m_\ell(r;\beta+1,(b+1)^{-1}) \frac{(b+2)^{-n-r}(\beta+1)_{n+r}}{(n! \ r!)} \tag{4.14}$$

$$= (\beta+1)_j (b^2+2b)^{\beta+1} \ j! \ \delta_{j,\ell} \ .$$

As a final example we consider the transform $(L,(-1)^n/(n!)^2)$ that maps $n^{(k)}$ to $x^k/k!$. The Charlier polynomials $c_j(x,a)$ are

defined by (see [34]

$$(4.15)\qquad c_j(x,a) = a^{j/2}(j!)^{-1/2} \sum_{k=o}^{j} (-1)^{j+k} \binom{j}{k}\binom{x}{k} k!\, a^{-k} \quad .$$

The simple Laguerre polynomials $L_j(x)$ are $L_j^{(0)}(x)$ of (4.12). Clearly

$$(4.16)\qquad L[c_j(n,a);(-1)^n/(n!)^2,x] = (-1)^j a^{j/2}(j!)^{-1/2} L_j(x/a) \quad .$$

By (4.2), and (4.13) and (4.16) we obtain

$$(4.17)\qquad \sum_{m,n=o}^{\infty} c_j(n,a)c_\ell(m,a)\theta_{m,n} = a^{j+1}\delta_{j,\ell}/j! \quad ,$$

where

$$\theta_{m,n} = a^{m+n} \sum_{r=o}^{\infty} \frac{(-a)^r}{r!} \binom{m+n+2r}{m+r} \quad .$$

Note that

$$L[c_j(n,a); \frac{(-1)^n}{n!(\beta+1)_n},x] = (-1)^j\, a^{j/2}\, (j!)^{-1/2}\, L_j^{(\beta)}(x/a) \quad ,$$

leads to a relation of the type (4.17); with $\theta_{m,n}$, of course, depending on α; but is rather complicated. The β_α operator associated with $\alpha_n = (-1)^n/(n!)^2$, say Θ, is

$$(4.18)\qquad \Theta f_n = (f_{n+1} - f_o)/(n+1) \quad .$$

Hence, by (3.6) and Rodrigues' formula for the simple Laguerre polynomials, namely

$$L_j(x) = e^x D^j x^j e^{-x} = (D-1)^j x^j \quad ,$$

we get

$$c_j(x,a) = (j!)^{1/2}\, a^{-3j/2}(1-a\theta)^j x^{(j)} . \tag{4.19}$$

Relations (4.5), (4.7), (4.8), (4.9), (4.14), (4.17) and (4.19) seem to be new.

In contrast with the Rice polynomials, both the Charlier and Meixner polynomials are also orthogonal, in the ordinary sense (see [15], vol. 2, pp. 225-226). Furthermore, both the Charlier and Meixner polynomials are preimages of Laguerre polynomials under (L,α) transforms. This suggests the following problem. Given an orthogonal polynomial set $\{p_j(x)\}_o^\infty$, determine all the orthogonal sets $\{Q_j(x)\}_o^\infty$ that are preimages of $\{p_j(x)\}_o^\infty$ under (L,α) transforms. The answer to this question in the case $\{p_j(x)\}_o^\infty$ are the Jacobi or Laguerre polynomials is given in Theorems 4.1 and 4.2 respectively.

Theorem 4.1. *The only orthogonal polynomial sets that are preimages of Jacobi polynomials* ([31], p. 263) *of argument* $1-2vx$ *under* (L,α) *transforms are* ${}_3F_2(-n,n+\gamma,x;\beta_1,\beta_2;1)$.

Proof. Let $\{Q_n(x)\}_o^\infty$ be such a set. Thus $Q_n(x)$ is of the form

$$Q_n(x) = \sum_{k=o}^{n} (-n)_k (n+\gamma)_k (x)_k\, \lambda_k , \tag{4.20}$$

where $\lambda_k \neq 0$ for $k = 0,1,\ldots$. Let

$$Q_{n+1}(x) = (A_n x+B_n)Q_n(x) + C_n\, Q_{n-1}(x)$$

be the three term recurrence relation satisfied by $Q_n(x)$, see [1]

and [32]. From (4.20) and the above three term recurrence relation we get

$$-(n+1)(n+\gamma+k)(n+\gamma+k-1) = (B_n-kA_n)(-n+k-1)(n+\gamma)(n+\gamma+k-1)$$
$$- C_n(-n+k)(-n+k-1)/n + A_n(n+\gamma)\lambda_{k-1}/\lambda_k \quad .$$

It is obvious from the above equation that

$$\frac{\lambda_{k-1}}{\lambda_k} = k^3+ak^2+bk+c \ , \qquad a,\ b,\ c \text{ are constants} \ ,$$

and where $A_n(x)$ is ${}_4F_3(-n,n+\gamma,x,1;\beta_1,\beta_2,\beta_3;1)$ and the rest follows by Theorem 4 of [1].

<u>Theorem 4.2.</u> <u>The only orthogonal polynomial sets that are preimages under</u> (L,α) <u>transforms of the Laguerre polynomials</u> $\{L_j^{(\beta)}(vx)\}_{j=o}^{\infty}$ <u>are the Meixner and Charlier polynomials.</u>

The proof is very similar to our proof of Theorem 4.1 and uses Theorem 3 of [1]. See also [20].

Note that Theorems 4.1 and 4.2 are also extensions of Theorems 4 and 3 of [1] respectively.

Now we come to preimages of the Hermite polynomials under (L,α) transforms. There is no orthogonal polynomial sets that belong to this class. In fact we have the more general result.

<u>Theorem 4.3.</u> <u>There is no orthogonal polynomial set of the type</u>

$$Q_n(x) = \sum_{k=o}^{[n/2]} \lambda_{n,k}(x)_{n-2k} \quad .$$

The proof follows immediately from the three term recurrence

relation.

Theorem 4.3 tells us precisely that the preimages of a simple set of symmetric polynomials never constitutes an orthogonal, in the generalized sense, set.

Now, it is clear that if we specify the orthogonal polynomial set $\{p_n(x)\}_0^\infty$ then the same question could be handled as we did for the Laguerre and Jacobi polynomials. However, the problem of determining all pairs of orthogonal polynomial sets $(\{P_j(x)\}_0^\infty, \{Q_j(x)\}_0^\infty)$ that satisfy $L[Q_j(n);\alpha,x] = P_j(x)$ for some (L,α) transform, seems to be much more general and remains to be investigated.

4.3 <u>Zeros of Polynomials Orthogonal with Respect to Convolution.</u> In the present section we restrict ourselves to sequences f_n for which $\sum \frac{x^n}{n!} f_n$ remains bounded on $(0,\infty)$. Thus definitions (3.1) and (3.4) are equivalent. Furthermore we shall assume that $\phi(x)$ is a Laplace transform of a non-negative function, that is

$$(4.21) \qquad \phi(x) = \int_0^\infty e^{-xt}\varepsilon(t)dt \quad \text{for} \quad x \in (0,\infty), \text{ with } \varepsilon(t) \geq 0 \text{ for } t \in (0,\infty) .$$

Therefore

$$(4.22) \qquad L[f;\alpha,x] = \int_0^\infty e^{-xt} \{ \sum_0^\infty \frac{(xt)^n}{n!} f_n \} \varepsilon(t) \, dt .$$

An operator T is variation diminishing if $V\{Tf\} \leq V\{f\}$, where $V\{f\}$ is the variation of f defined as the number of sign changes of the function as x varies across its domain [18].

Lemma 1. Let $f(x) = \sum_{0}^{\infty} a_k x^k$ be uniformly convergent on $[0,a]$, then on $[0,a]$, $V[f] \leq V[\{a_k\}]$, where $V[\{a_k\}]$, is the number of sign changes in the sequence $a_0, a_1, \ldots$.

Lemma 2. If $f(x) = \int_{0}^{\infty} e^{-xt} g(t)dt$, then on $[0,\infty)$, $V[f] \leq V[g]$.

Lemma 1 is due to Cheney and Sharma [12]. It follows easily from Descartes rule of signs. Lemma 2 follows from Theorems 7.1 and 9.1a of [18] on pages 97 and 103, respectively.

We now come to the main result in this section.

Theorem 4.4. *Suppose that $\{p_j(x)\}_{j=o}^{\infty}$ is a polynomial set orthogonal on a subset of $[0,\infty)$ and $Q_j(x) = L^{-1}[p_j(x); \alpha, n]$ with $\phi(x) = \sum \alpha_n x^n$ satisfying (4.21). Then $Q_j(x)$ has real and simple zeros.*

Proof. Lemmas 1 and 2, and (4.22) imply that the number of sign changes of $p_j(x)$ in $[0,\infty)$ is at most $V[\{Q_j(n)\}_{n=o}^{\infty}]$. On the other hand $p_j(x)$ has j changes of sign in $[0,\infty)$. Thus, j sign changes of $Q_j(x)$ must occur at $x = 0, 1, \ldots$ and the result follows since $Q_j(x)$ is a polynomial of degree j.

Corollary 1. The zeros of the Rice polynomials are real and simple for $p > 0$.

Proof. One can argue that

$$L[H_j(-n,p,v); \frac{(p)_n}{n!}, x] = P_j(1-2vx) \quad ,$$

$P_j(x)$ being the Legendre polynomial of order j. For $v > 0$, the Legendre polynomials of argument $1-2vx$ are orthogonal on $[0, \frac{1}{v}]$.

The corresponding $\phi(x)$ is $(1+x)^{-p}$ which is the Laplace transform of $e^{-t}\, t^{p-1}/\Gamma(p)$. Therefore $H_j(x;p,v)$ has real and simple zeros for $v > 0$ and $p > 0$. Let

$$H_j(x,p,-v) = \frac{(-v)^j}{(p)_j} \binom{2j}{j} \prod_{k=1}^{j} (x-\theta_{k,p,v}) , \quad v > 0 ,$$

where the θ's are distinct. Clearly the θ's are real polynomials in v^{-1}. Therefore $H_j(x,p,v)$ also has real and simple zeros.

<u>Corollary 2</u>. If $\{Q_j(x)\}_{j=o}^{\infty}$ are orthogonal with respect to convolution and the points of increase of the distribution ψ of (4.2) are contained in $(0,\infty)$, then the zeros of $Q_j(x)$ are real and simple.

CHAPTER V

MISCELLANEOUS RESULTS

This chapter contains several miscellaneous results related to the subject of this thesis.

Recall that the theory developed in Chapter III enables us to handle difference equations of the type

$$(5.1) \qquad \sum_{k=o}^{m} a_k \Gamma^k f(n) = g(n), \qquad a_0, a_1, \ldots, a_m \text{ are constants,}$$

for operators Γ of a particular type. Consequently one would like to have a way of recognizing difference equations of the above type. In other words one needs some operational formulas to expand powers of Γ in terms of the shift operator E and its inverse E^{-1} in order to be able to write a given difference equation in the form (5.1), if possible. So, we devote the first section, 5.1, to expansion formula of $(\Delta x \nabla)^n$ in terms of powers of E and E^{-1} and illustrate how to obtain such formulas for more general Γ's. Osipov ([27], [28]) studied the problem of expanding the Bessel operator DxD and its generalization $x^{-\gamma} D x^{\gamma+1} D$ in powers of D. Carlitz [11] independently gave a similar expansion for the Bessel operator.

In §5.2 we introduce a q-analogue of the Bessel operator DxD, namely the operator $D_q xD_q$ where

$$(5.2) \qquad D_q f(x) = \{f(qx) - f(x)\}/x(q-1) \quad .$$

As we shall see these considerations will lead to a q-analogue of the Laplace-Carson transform. Some applications related to Heine series and to q-difference equations will be mentioned. It is clear that the differentiation operator D is the limiting case of D_q as $q \to 1$.

Finally in §5.3 we shall indicate some possible extensions to multi-dimensional cases. In particilar we shall indicate a method of solving some mixed equations.

5.1 Expansion Formulas. Recall that $\beta = \Delta x \nabla$ satisfies

$$\beta f(x) = (x+1)f(x+1) - (2x+1)f(x) + xf(x-1) \quad .$$

It is clear that

$$(5.3) \qquad \beta^n = \sum_{k=-n}^{n} (-1)^{k+n} a_{n,k}(x) E^k \quad ,$$

that is

$$(5.4) \qquad \beta^n f(x) = \sum_{k=-n}^{n} (-1)^{k+n} a_{n,k}(x) f(x+k) \quad ,$$

where the $a_{n,k}(x)$'s are polynomials in x of degree at most n. The relation

$$(5.5) \qquad \sum_{o}^{r} (-1)^{r+j} \binom{r}{j} \binom{j}{s} = \delta_{r,s} \quad ,$$

for non-negative integers r and s, $r \geq s$, follows easily from the binomial theorem.

Let $f(m) = \binom{m}{j}$ in (5.4), for positive integers m, to get

$$(5.6) \qquad \frac{j!}{(j-n)!}\binom{m}{j-1} = \sum_k (-1)^{k+n} a_{n,k}(m) \binom{m+k}{j} \quad .$$

Multiplying (5.6) by $(-1)^{m+\ell+j}\binom{j}{m+\ell}$ and summing over all j, by (5.5), we obtain

$$(5.7) \qquad (-1)^n a_{n,\ell}(m) = n! \sum_j (-1)^{j+m} \binom{j}{n}\binom{m}{j-n}\binom{j}{m+\ell} \quad ,$$

$$\ell = 0, \pm 1, \ldots, \pm n \ .$$

In the left hand side of (5.7), j runs from $\min\{n, m+\ell\}$ to $m+n$. Replace j by $j+m+\ell$ in (5.7) in order to have

$$(5.8) \qquad a_{n,\ell}(m) = n! \sum_j (-1)^{j+n+\ell} \binom{j+m+\ell}{n}\binom{m}{n-\ell-j}\binom{j+m+\ell}{j} \quad .$$

Now, for fixed n and ℓ, the polynomials $(-1)^{\ell} a_{n,\ell}(x)/n!$ and $\sum_j (-1)^{j+n}\binom{j+\ell+x}{n}\binom{x}{n-\ell-j}\binom{x+j+\ell}{j}$ agree at all positive integers, hence they are identical, that is

$$(5.9) \qquad a_{n,\ell}(x) = n!\,(-1)^{\ell+n} \sum_{j=o}^{n-\ell} (-1)^j \binom{x+j+\ell}{n}\binom{x}{n-\ell-j}\binom{x+j+\ell}{j} \ ,$$

where

$$\binom{x}{j} = x(x-1)\ \ldots\ (x-j+1)/j! \quad .$$

Clearly the above procedure can be duplicated for any operator Γ satisfying

$$\Gamma\binom{m}{k} = c_k \binom{m}{k-1} \quad , \quad k = 0,1,\ldots \quad .$$

Indeed let

$$(5.10)\qquad \Gamma^n f(x) = \sum_{k=-\infty}^{\infty} (-1)^{k+n}\, a_{n,k}(x)\, E^k f(x) \quad .$$

For $f(m) = \binom{m}{j}$ we get

$$(5.11)\qquad c_j \cdot c_{j-1} \cdots c_{j-n+1} \binom{m}{j-n} = \sum_{k=-\infty}^{\infty} (-1)^{k+n} a_{n,k}(m) \binom{m+k}{j} \quad .$$

Multiply (5.11) by $(-1)^{m+\ell+j}\binom{j}{m+\ell}$ and sum over j to get, by (5.5)

$$(5.12)\qquad a_{n,\ell}(m) = \sum_j (-1)^{j+n+\ell} \binom{j+m+\ell}{j} \binom{m}{n-\ell-j}\, c_{j+m+\ell} \cdots c_{j+m+\ell-n+1} \quad .$$

We see that if $c_k = k$, i.e., $\Gamma = \beta$, formula (5.12) reduces to (5.8). In the case $\Gamma = \beta_\gamma$ of Chapter III we have

$$c_k = (\gamma+k) \ , \qquad k > 0 \ ,$$

and

$$(5.13)\qquad a_{n,\ell}(m) = n! \sum_j (-1)^{j+n+\ell} \binom{j+m+\ell}{j} \binom{m}{n-\ell-j} \binom{j+m+\ell+\gamma}{n} \quad .$$

Clearly (5.13) reduces to (5.8) when $\gamma = 0$.

Now we go back to (5.9) to write $a_{n,\ell}(x)$ in a more compact form. Clearly for $k \geq 0$ we have

$$(5.14)\qquad a_{n,-k}(x) = n!(-1)^{n+k} \sum_{j=o}^{n+k} (-1)^j \binom{j-k+x}{n} \binom{x}{n+k-j} \binom{x+j-k}{j} \ .$$

On the other hand

$$\binom{x}{n+k-j} \binom{x+j-k}{j} = \binom{n+k}{j} x^{(k)} (x+j-k)^{(n)} / (n+k)! \quad ,$$

and (5.14) imply

$$(5.15)\qquad a_{n,-k}(x) = x^{(k)} \sum_{j=o}^{n+k} (-1)^{n+k+j} \binom{n+k}{j} (x+j-k)^{(n)}, \quad k= 0,1,\ldots ,$$

that is

$$(5.16)\qquad a_{n,-k}(x) = \frac{x^{(k)}}{(n+k)!} \Delta^{n+k} \{(x-k)^{(n)}\}^2 , \qquad k = 0,1,\ldots,n .$$

Furthermore, we have

$$(5.17)\qquad a_{n,k}(x) = \sum_{j=o}^{n-k} \frac{(-1)^{n-k-j}}{n!(n-k-j)!} \frac{\{(x+j+k)^{(n)}\}^2}{(x+1)_k} , \quad k \geq 0 .$$

Relation (5.17) may be written as

$$(5.18)\qquad a_{n,k}(x) = [(n-k)!(x+1)_k]^{-1} \Delta^{n-k}\{(x+k)^{(n)}\}^2 , \qquad k = 0,1,\ldots,n .$$

Going back to (5.3) and using $\beta^{n+1} = \beta\ \beta^n$ we obtain the recurrence relation

$$(5.19)\qquad a_{n+1,k}(x) = (x+1)a_{n,k-1}(x) + (2x+1)a_{n,k}(x) + xa_{n,k+1}(x-1) .$$

Set

$$(5.20)\qquad a_{n,k}(x) = (x+1)_k b_{n,k}(x),\ a_{n,-k}(x) = x^{(k)} b_{n,-k} , \qquad k = 0,1,\ldots,n .$$

One can easily see from (5.19) that $b_{n,-k}(x)$ satisfies, for $k \geq 1$, the relation

$$(5.21)\qquad b_{n+1,-k}(x) = (x+1)^2 b_{n,-k-1}(x)+(2x+1)b_{n,-k}+b_{n,-k+1}(x-1) \quad .$$

From (5.9) and (5.16) one can easily see that $b_{n,k}(x-k)$ also satisfies (5.21). Thus $b_{n,-k}(x)$ and $b_{n,k}(x-k)$ must be identical since $b_{n,-n}(x) = b_{n,n}(x-n) = 1$. Consequently

$$(5.22)\qquad a_{n,k}(x) = \frac{(x+1)_k}{(k+n)!}\,\Delta^{n+k}\{x^{(n)}\}^2 \quad , \quad k = 0,1,\ldots,n \quad ,$$

and

$$(5.23)\qquad a_{n,-k}(x) = [(n-k)!(x)_k]^{-1}\,\Delta^{n-k}\{x^{(n)}\}^2 \quad , \quad k = 0,1,\ldots,n \quad .$$

This leads to the curious identity

$$(5.24)\qquad \Delta^{n-k}\{(x+k)^{(n)}\}^2 = \{(x+1)_k\}^2(n-k)!\,\Delta^{n+k}\{x^{(n)}\}^2 \quad , \quad k = 0,1,\ldots,n.$$

5.2 q-Analogue

5.2.1 q-Analogue of the Bessel Operator.

The q-difference operator D_q is defined by

$$D_q f(x) = \{f(x)-f(qx)\}/x(1-q) \quad .$$

It is clear that, for differentiable functions, the differentiation operator $\frac{d}{dx}$ is the limiting case of D_q, as $q \to 1$. We define a q-analogue of the Bessel operator by means

$$(5.25)\qquad B_q f(x) = (D_q x D_q)f(x) = \frac{f(x)-2f(qx)+f(q^2x)}{(1-q)^2 x} \quad .$$

We begin by introducing some notations. q-factorials $[n]!$ and q-binomial coefficients $\left[{n \atop k}\right]$ are defined by

(5.26) $$[n]! = (q;n)/(1-q)^n ,$$

where

(5.27) $$(q;n) = \prod_{i=1}^{n} (1-q^i) , \quad n > 0 \quad \text{and} \quad (q;0) = 1 ,$$

and

(5.28) $$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[k]![n-k]!} = \begin{bmatrix} n \\ n-k \end{bmatrix} .$$

Moreover, by $[a]$ we shall always mean $(q^a-1)/(q-1)$. We shall assume $0 < q < 1$.

There are two q-analogues of the exponential function. Indeed the exponential function e^x is the limit as $q \to 1$ of both $e_q((1-q)x)$ and $E_q((1-q)x)$, where

(5.29) $$e_q(x) = \prod_{j=o}^{\infty} (1-xq^j)^{-1} = \sum_{j=o}^{\infty} x^j/(q;j) ,$$

and

(5.30) $$E_q(x) = \prod_{j=o}^{\infty} (1+xq^j) = \sum_{j=o}^{\infty} q^{j(j-1)/2} x^j/(q;j) .$$

We shall also use [17]

(5.31) $$(a+b)_\nu = a^\nu \prod_{j=o}^{\infty} \left(\frac{1+\frac{b}{a}q^j}{1+\frac{b}{a}q^{j+\nu}}\right) = a^\nu \sum_{n=o}^{\infty} \frac{(1-q^{-\nu})_n}{(q;n)} \left(-\frac{bq^\nu}{a}\right)^n .$$

In particular if $\nu = \pm N$, N is a positive integer, formula (5.31) becomes

$$(5.32)\qquad (a+b)(a+bq)\ldots(a+bq^{N-1}) = \sum_{n=o}^{N} [{N \atop n}] q^{n(n-1)/2} b^n a^{N-n} \quad ,$$

and

$$(5.33)\qquad \frac{a^N}{(a+bq^{-1})\ldots(a+bq)^{-N}} = \sum_{n=o}^{\infty} [{n+N-1 \atop n}] (-\frac{bq^{-N}}{a})^n$$

respectively.

We are interested in finding a transform, L_q say, and a convolution produce $*$ such that

$$(5.34)\qquad L_q[f*g;x] = L_q[f;x]L_q[g;x] \quad ,$$

holds and L_q maps B_q to D_q, that is

$$(5.35)\qquad L_q[B_q f;x] = D_q L_q[f;x] \quad .$$

In this setting, one might expect that

$$(5.36)\qquad L_q[t^n/[n]!;x] = x^k \quad .$$

This is so since $B_q \frac{x^n}{[n]!} = [n] \frac{x^{n-1}}{[n-1]!}$, while $D_q x^n = [n]x^{n-1}$. The above interlation between $*$, L_q, B_q and D_q resembles the interlation between the convolution of Chapter III, the (L,α_n) transform, the operator β_α and the differentiation operator D respectively.

We recall that the q-analogue of the Laplace transform (see [17])

$$ {}_q\ell_s f = s^{-1}E_q(-q) \sum_{n=o}^{\infty} \frac{q^n}{(q;n)} f(q^n/s) \quad ,$$

satisfies

$$\ell_{q}{}_{s}\, t^{n} = s^{-n-1}(q;n) \qquad (n = 0,1,\ldots) \;,$$

which suggests that we consider, as the q-analogue of the L transform in Chapter II, the transform L_q defined as

(5.37) $$L_q[f;x] = E_q(-q) \sum_{0}^{\infty} q^j\, f(\frac{xq^j}{1-q})/(q;j) \quad .$$

It is easy to see that (5.34) as well as

(5.38) $$L_q[D_q f;x] = \{L_q[f;x]-L_q[f;0]\}/x \quad ,$$

are valid under (5.37).

Furthermore we have

Theorem 5.2.1. The relations

(5.39) $$L_q[t^n f(t);x] = (xD_q x)^n L_q[f;n]$$

and

(5.40) $$L_q[(t\,\frac{d}{dt})^n f(t);x] = (x\,\frac{d}{dx})^n L_q[f;x]$$

and (5.35) holds.

Proof. We have

$$xD_q\{xL_q[f;x]\} = \frac{E_q(-q)}{1-q} \sum_{n=o}^{\infty} \frac{q^n}{(q;n)}\, \{xf(\frac{xq^n}{1-q})-qxf(\frac{xq^{n+1}}{1-q})\}$$

$$= \frac{x}{1-q}\, E_q(-q)\, \{f(\frac{x}{1-q}) + \sum_{1}^{\infty} q^{2n}\, f(\frac{xq^n}{1-q})/(q;n)\}$$

$$= E_q(-q) \sum_{0}^{\infty} \frac{q^n}{(q;n)}\, (\frac{xq^n}{1-q}) f(\frac{xq^n}{1-q}) = L_q[tf(t);x] \quad ,$$

and (5.39) follows for $n = 1$, and by induction for all n. Formula (5.40) is obvious and (5.35) follows from (5.38) and (5.39).

Let L_q^{-1} be the inverse of L_q. We have (see [17])

$$(5.41) \qquad L_q^{-1}[f;x] = e_q(q) \sum_{o}^{\infty} \frac{(-1)^j q^{j(j+1)/2}}{(q,j)} f(xq^j(1-q)) \quad .$$

Note that (5.37) as well as (5.41) are valid at least for functions that are regular at the origin.

We are now in a position to introduce a convolution product $*$ so that

$$(5.42) \qquad L_q[f*g;x] = L_q[f;x]L_q[g;x] \quad .$$

Or, in other words

$$(f*g)(x) = L_q^{-1}\{L_q[f;x]L_q[g;x];x\} \quad .$$

By (5.37) and (5.41) we therefore must have

$$(5.43) \qquad (f*g)(x) = E_q(-q) \sum_{o}^{\infty} \frac{(-1)^j \, q^{j(j+1)/2}}{(q;j)} \sum_{m,n=o}^{\infty} \times q^{m+n} \frac{f(xq^{m+j})g(xq^{n+j})}{(q;m)(q;n)} \quad .$$

Now having the definition (5.43) at hand we can verify (5.42) directly. The set of all functions that are regular at the origin forms a commutative ring, under addition and convolution (5.43). This ring has no zero divisors. One can easily construct an operational calculus based on the convolution (5.43), as we did in Chapter III. The technique is very similar to that of Chapter III

and we shall leave it out.

5.2.2 Application to Functional Equations. First of all, the linear q-difference equation with constant coefficients is transformed under L_q to an algebraic equation and its treatment is identical with our treatment of the linear difference equation with constant coefficients in Chapter II.

Our second application is to q-difference equations of the type

$$(5.44) \qquad \sum_{j=o}^{n} a_j (xD_q x)^j F(x) = G(x) \quad ,$$

where the a's are constants. Let $f(x) = L_q^{-1}[F;x]$ and $g(x) = L_q^{-1}[G;x]$. Equation (5.44) is therefore equivalent to, by (5.34)

$$\left(\sum_{o}^{n} a_i t^i\right) f(t) = g(t) \quad ,$$

and hence

$$F(x) = L_q\left[\frac{L_q^{-1}[g;t]}{\sum_{o}^{n} a^i t^i} ;x\right] \quad .$$

The limiting case, as $q \to 1$, of (5.44) can be treated similarly using the Carson-Laplace transform.

Our third application is to q-difference - differential equations with constant coefficients in D_q and $x \frac{d}{dx}$. Such an equation, say

$$\sum_{i=o}^{n} a_i (x \frac{d}{dx})^i f(x) + \sum_{j=1}^{m} b_j D_q^j f(x) = g(x)$$

is transformed under L_q to the much simpler equation

$$\sum_{i=o}^{n} a_i (x \frac{d}{dx})^i F(x) + (\sum_{j=1}^{m} b_j x^{-j}) F(x)$$

$$- \sum_{j=1}^{m} \sum_{r=o}^{j-1} x^{r-j} F^{(r)}(0) = G(x) ,$$

$F(x)$ and $G(x)$ being the L_q transform of f and g respectively.

As an example of the above type consider the functional equation

$$(q-1)x^2 f'(x)+f(qx)-f(x) = \frac{x^2(q-1)}{2} + x^3(q-1)/(1+q) ,$$

whose L_q transform is

$$xF'(x) + \frac{F(x)-F(0)}{x} = x^2 + x/2 ,$$

where $F(x)$ is as before. Thus

$$F(x) = F(0) + \frac{1}{2} x^2 ,$$

or

$$f(x) = f(0) + \frac{x^2}{2(1+q)} .$$

5.2.3 Applications to Basic Hypergeometric Functions. A basic hypergeometric function ${}_m\phi_n$ is defined by the Heine series

$$ {}_m\phi_n(a_1,\ldots,a_m;b_1,\ldots,b_n;x) = \sum_{r=0}^{\infty} \frac{(1-a_1)_r \ldots (1-a_m)_r}{(1-b_1)_r \ldots (1-b_n)_r} x^r/(q;r) \quad . $$

These functions are q-analogues of the generalized hypergeometric functions ${}_mF_n$.

Meller [22] and [23] used the following method to obtain identities among special functions. Let T_1 and T_2 be two given operators and suppose that one can find an invertible operator U satisfying $UT_1 = T_2U$ and $U1 = 1$, where 1 is the constant function $f(x) = 1$. By induction we have $UT_1^n = T_2^nU$, $n = 0, \pm 1, \pm 2$. Therefore

$$ (5.45) \qquad U(\sum_{o}^{\infty} a_n T_2^{-n})1 = (\sum a_n T_2^{-n})U1 = (\sum a_n T_2^{-n})1 \; . $$

In general (5.45) and

$$ (5.46) \qquad (\sum a_n T_1^{-n})1 = U^{-1}(\sum a_n T_2^{-n})1 \; , $$

express identities. If we take for T_1, T_2, q-difference operators we get identities involving basic hypergeometric series. In particular, put $a_n = a^n$, that is

$$ \sum_{o}^{\infty} a_n T^n = \frac{T}{T-a} \quad . $$

<u>Case I</u>. Take $T_1 = D_q$ and $T_2 = D_q x^{-\beta} D_q x^{\beta+1}$. In this case it is easy to see that $Ux^n = (1-q)^n ((1-q)_\alpha/(1-q)_{\alpha+n}) x^n$, so that U may be defined by

$$(5.47)\qquad Uf(x) = e_q(q^{\beta+1}) \sum_{j=o}^{\infty} \frac{(-q^{\beta})^j \, q^{j(j+1)/2}}{(q;j)} f(xq^j(1-q)) \quad ,$$

and one can check directly to see that the operator U defined by (5.47) satisfies all the requirements. Furthermore, we have

$$U^{-1}f(x) = E_q(-q^{\beta+1}) \sum_{o}^{\infty} \frac{(-q^{\beta+1})^j}{(q;j)} f(\frac{xq^j}{1-q}) \quad ,$$

$$\frac{T_1}{T_1-a} 1 = e_q((1-q)ax) \qquad \text{and} \qquad \frac{T_2}{T_2-a} 1 = {}_0\phi_1(-;\beta+1;(1-q)^2ax) \quad .$$

From (5.45) and (5.46) we get

$$e_q(q^{\beta+1}) \sum_{j=o}^{\infty} (-q^{\beta})^j \frac{q^{j(j+1)/2}}{(q;j)} e_q(xq^j) = {}_0\phi_1(-;\beta+1;x)$$

and

$$e_q(x) = E_q(-q^{\beta+1}) \sum_{o}^{\infty} \frac{(-q^{\beta+1})^j}{(q;j)} {}_0\phi_1(-;\beta+1;xq^j) \quad .$$

<u>Case II</u>. Take $T_1 = D_2$ and $T_2 = x^{-\beta-1} D_q x^{\beta+1}$, $\neq 0,-1,-2,\ldots$. As in Case I we first establish

$$Ux^n = \frac{(1-q)_{\beta}(q;n)}{(1-q)_{\beta+n}} x^n \quad , \qquad n = 0,1,\ldots \quad ,$$

and try to guess the definition of U. Note that the operator U of Case I maps x^n to $((1-q)^n(1-q)_n)/((1-q)_{\beta+n})\, x^n$ and L_q maps x^n to $(q,n)x^n/(1-q)^n$. Thus it is conceivable that the composition of these operators is the required operator. This is indeed so and we set

$$Uf(x) = \frac{E_q(-q)e_q(q^\beta)}{(1-q)_{\beta-1}} \sum_0^\infty \frac{q^j(1-q)_{\beta+j-1}}{(q;j)} f(xq^j) \quad ,$$

and hence

$$U^{-1}f(x) = \frac{E_q(-q^{\beta+1})e_q(q)}{(1-q)_{\beta+1}} \sum_0^\infty \frac{(1-q)_{\beta+j+1}}{(q;j)} f(xq^j) \quad .$$

In this case,

$$\frac{T_1}{T_1-a} 1 = e_q((1-q)ax) \quad \text{and} \quad \frac{T_2}{T_2-a} 1 = {}_1\phi_1(1;\beta+1;(1-q)ax) \quad ,$$

and (5.45) and (5.46) imply

$$\frac{E_q(-q)e_q(q^{\beta+1})}{(1-q)_{\beta-1}} \sum_0^\infty q^j \frac{(1-q)_{\beta+j-1}}{(q;j)} e_q(xq^j) = {}_1\phi_1(1;\beta+1;x) \quad ,$$

and

$$\frac{E_q(-q^{\beta+1})e_q(q)}{(1-q)_{\beta+1}} \sum_0^\infty q^j \frac{(1-q)_{\beta+j+1}}{(q;j)} {}_1\phi_1(1;\beta+1;xq^j) = e_q(x) \quad .$$

It is clear that we can repeat this process and get many more identities. For more results of this type regarding ordinary hypergeometric functions see [22] and [23].

5.2.4 <u>Another q-analogue</u>. The L_q transform of the previous section, being a function to function transform, might not be a proper analogue of any (L,α) transform. However, one can define such a q-analogue by

$$(5.48) \qquad \Lambda_q[f;\alpha,x] = \sum_0^\infty \frac{(-x)^n}{[n]!} q^{-n(n-1)} f(n) D_q^n \phi(xq^{-n}) \ , \quad q \geq 1 \ ,$$

where α stands for the sequence $\{\alpha_n\}_o^\infty$, and $\phi(x) = \sum_o^\infty \alpha_n x^n$. By $D_q^n \phi(xq^{-n})$ we mean, of course, $D_q^n \phi$ evaluated at xq^{-n}. In the present section we shall restrict ourselves to $q > 1$, and $q = 1$ will be a limiting case. If $\phi(x)$ has a positive radius of convergence then

$$(5.47) \qquad \Lambda_q[f;\alpha,x] = \sum_{s=o}^{\infty} \alpha_s x^s \sum_{n=o}^{s} (-1)^n [{}^s_n] w^{n(n-1)/2} f(n) q^{n(1-s)} \quad ,$$

for x belonging to some neighborhood of the origin. We shall assume that $\alpha_n \neq 0$, $n = 0,1,\ldots,$ in order to make Λ_q ont-to-one. Furthermore we suppose $\alpha_o = 1$. Let

$$(5.50) \qquad \theta_j(n) = q^{j(j-1)/2}[{}^n_j] \quad , \quad \xi_j(n) = (-1)^j \theta_j(n)/\alpha_j \quad ,$$

and

$$(5.51) \qquad c_o = 0 \quad \text{and} \quad c_{j+1} = -\alpha_{i+1}/\alpha_j \quad , \quad j \geq 0 \quad .$$

It is easy to see that

$$(5.52) \qquad f(n) = \sum_{j=o}^{n} b_j(f)\ \theta_j(n) \quad , \quad (n = 0,1,\ldots) \quad ,$$

where

$$(5.53) \qquad b_j(f) = (q^{-\ell}\Delta)^j\ f(\ell)\Big|_{\ell=o} \quad .$$

Formula (5.53) may also be written as

$$(5.54) \qquad b_j(f) = \sum_{r=o}^{j} (-1)^{j+r}\ [{}^j_r]\ q^{r(r+1)/2}\ q^{-jr}\ f(r) \quad ,$$

and symbolicially, by (5.32), as

$$b_j(f) = q^{-j(j-1)/2} (E-1)(E-q)\ldots(E-q^{j-1}) f(0) \quad .$$

Now substituting for $f(n)$ from (5.52) in (5.49) and using (5.32) we get

$$(5.55) \qquad \Lambda_q[f;\alpha,x] = \sum_{s=o}^{\infty} (-x)^s \alpha_s b_s(f)$$

$$= \sum_{s=o}^{\infty} (-x)^s \alpha_s q^{-s(s-1)/2} (E-1)(E-q)\ldots)E-q^{s-1})f(0) ,$$

which may be compared with (3.4). We shall adopt (5.55) as our definition of the $\Lambda_q[f;\alpha,x]$, or (Λ_q,α), tranform.

We point out here that (5.55) may also be written formally as

$$\Lambda_q[f;\alpha,x] = \sum_{r-o}^{\infty} (-x)^r q^{-r(r-1)/2} f(r) \sum_{j=o}^{\infty} x^j [{}^{j+r}_{\ r}]q^{-jr} \alpha_{j+r}$$

$$= \{ \sum_{r=o}^{\infty} f(r) \frac{(-qxE)^r}{(1-xE)(q-xE)\ldots(q^r-xE)} \}\alpha_o \quad ,$$

by (5.33). Therefore we get the formal relation

$$\Lambda_q[f;\alpha,x] = \Lambda_q[f;(-1)^n,-xE] \alpha_o \quad ,$$

which in turn may be compared with a similar formula in Chapter III.

The case $\alpha_n = (-1)^n$ is of special interest. We have, for $|x| < 1$,

$$(5.56) \qquad \Lambda_q[f;(-1)^n,x] = \sum_{n=o}^{\infty} \frac{(qx)^n f(x)}{(1+x)(q+x)\ldots(q^n+x)} \quad .$$

In case $\alpha_n = \dfrac{(-1)^n}{[n]!}$ we have

$$\Lambda_q[f;\frac{(-1)^n}{[n]!}, x] = e_q((q-1)x) \sum_{r=o}^{\infty} \frac{(x)^r}{[r]!\ ((1-q^{-1})x+1)_r} f(r) \quad .$$

Or if $\alpha_n = \frac{(-1)^n}{[n]!} q^{n(n-1)/2}$, we get

$$\Lambda_q[f;\ \frac{(-1)^n q^{n(n-1)/2}}{[n]!}, x] = E_q(x(q-1)) \sum_{r=o}^{\infty} \frac{x^r\ f(r)}{[n]!} \quad .$$

On the other hand, corresponding to (3.36) we have $\alpha_n = (-1)^n[{n+\alpha \atop n}]$ and

$$\Lambda_q[f;(-1)^n[{\alpha+n \atop n}],x] = \sum_{r=o}^{\infty} x^r(1+xq^{\alpha+1})_{-r-\alpha-1}\ q^{-r(r-1)/2}\ f(r)$$

$$= \sum_{r=o}^{\infty} x^r \frac{q^{-r(r-1)/2}}{(1+xq^{-r})_{\alpha+r+1}} [{\alpha+r \atop r}]\ f(r) \quad .$$

Let us define the operators $A_{\alpha,q}$ and $B_{\alpha,q}$, that are q-analogoues of A_α and B_α of Chapter III, by

(5.57) $$A_{\alpha,q}\{f(n)\} = \sum_{j=o}^{n} b_{j+1}(f)c_{j+1}\theta_j(n),$$

and

(5.58) $$B_{\alpha,q}\{f(n)\} = -\sum_{j=o}^{n} b_{j+1}(f)[-j-1]c_{j+1}\theta_j(n) \quad .$$

The relations

(5.59) $$B_{\alpha,q} = -A_{\alpha,q}[-n]\nabla \quad ,$$

and,

(5.60) $$A_{\alpha,q}\theta_j(n) = c_j\theta_{j-1}(n) \quad \text{and} \quad B_{\alpha,q}\theta_j(n) = -[-j]c_j\theta_{j-1}(n) \quad .$$

are obvious.

Theorem 5.2. We have

(5.61) $$\Lambda_q[\xi_j(n);\alpha,x] = x^j \qquad j = 0,1,\ldots$$

(5.62) $$\Lambda_q[A_{\alpha,q}f;\alpha,x] = \frac{F(x)-F(0)}{x}$$

and

(5.63) $$\Lambda_q[A_{\alpha,q}f;\alpha,x] = D_{1/q}\, F(x) \quad ,$$

where $F(x) = \Lambda_q[f;\alpha,x]$.

Proof. Relation (5.60) clearly follows from (5.55) and (5.52). The definitions (5.57) and (5.58) tell us precisely that $b_j(A_{\alpha,q}f) = c_{j+1}\, b_{j+1}(f)$ and $b_j(B_{\alpha,q}f) = -[-j-1]c_{j+1}\, b_{j+1}(f)$ respectively. This obviously proves (5.62) and (5.63).

Thus a linear equation with constant coefficients in $A_{\alpha,q}$ is transformed under (Λ_q,α) to an algebraic equation.

We define the convolution product by

(5.64) $$\alpha_m b_m(f*g) = \sum_{r=o}^{m} \alpha_r\, \alpha_{m-r}\, b_r(f)b_{m-r}(g) \quad ,$$

which implies

(5.65) $$\Lambda_q[f*g;\alpha,x] = \Lambda_q[f;\alpha,x]\, \Lambda_q[g;\alpha,x] \quad .$$

In case $\alpha_n = (-1)^n$, $(-1)^n[{\gamma+n \atop n}]$ or $\frac{1}{[-j]!}$, the $B_{\alpha,q}$ operator will be $-q^{-n}\Delta\,[-n]\nabla$, $q^{-n}\Delta\,([\gamma]-[-n]\nabla)$ and $q^{-n}\Delta$ respectively. Furthermore $q^{-n}\Delta$ will be the $A_{\alpha,q}$ operator when $\alpha_n = (-1)^n$. Note that $-q^{-n}\Delta\,[-n]\nabla$ may be written as

$$-q^{-n}\Delta[-n]\nabla f = q^{-n}\{-[-n-1f(n+1)+([-n]+[-n-1])f(n)-[-n]f(n-1)\} .$$

In the rest of this section we shall denote $\Lambda_q[f;(-1)^n,x]$ by $\Lambda_q[f;x]$ or $F(x)$. Recall that $\Lambda_q[f;(-1)^n,x]$ is defined by (5.56). Clearly, as $q \to 1$, $\Lambda_q[f;x]$, $q^{-n}\Delta$ and $-q^{-n}\Delta[-n]\nabla$ reduce to $L[f;x]$. Δ and β of Chapter II. It is easy to deduce

$$(5.66) \qquad \Lambda_q[f(n+1);x] = F(qx) + \{F(qx)-F(0)\}/qx \quad ,$$

and

$$(5.67) \qquad \Lambda_q[f(n-1);x] = \frac{x}{x+1} F(x/q) \quad ,$$

from (5.56).

Theorem 5.3. The following formulas

$$(5.68) \qquad \Lambda_q[q^{-n}\nabla f;x] = F(x/q)/(x+1) \quad ,$$

$$(5.69) \qquad \Lambda_q[q^n f;x] = (1+xq)F(xq)-xF(x) \quad ,$$

and

$$(5.70) \qquad \Lambda_q[[n]^k f;x] = \{xD_q(1+x)\}^k F(x), \quad k = 1,2,\ldots ,$$

are valid.

Proof.

$$\Lambda_q[q^{-n}\nabla f;x] = \sum_o^\infty \frac{x^n}{(x+1)\ldots(x+q^n)} \{f(n)-f(n-1)\} = \sum_o^\infty \frac{x^n(x+q^{n+1}-x)}{(x+1)\ldots(x+q^{n+1})} f(n)$$

$$= (1+x)^{-1} \sum_o^\infty \frac{x^n}{(\frac{x}{q}+1)\ldots(\frac{x}{q}+q^n)} f(n) \quad ,$$

and (5.68) follows. Clearly we have

$$F(qx) = (1+qx)^{-1} \sum_{o}^{\infty} \frac{(xq)^n}{(x+1)\dots(x+q^{n-1})} f(n)$$

$$= (1+xq)^{-1} \sum_{o}^{\infty} \frac{(xq)^n (x+q^n)}{(x+1)\dots(x+q^n)} f(n)$$

$$= (1+qx)^{-1} \{xF(x)+\Lambda_q[q^n f(n);x]\} ,$$

which proves (5.69). Formula (5.70), for $k = 1$, follows from (5.69) and can be established for general k by easy induction.

We now proceed to evaluate the Λ_q transforms of E^k and E^{-k}, E being the shift operator. By iterating (5.67) we get

(5.71) $$\Lambda_q[E^{-k}f;x] = \frac{x^k}{(x+1)_k} F(xq^{-k}) .$$

To evaluate $\Lambda_q[E^k f;x]$, let $E^k f = g$, that is $f = E^{-k}g+\{f_0,f_1,\dots,f_{k-1},0,0,\dots\}$. Thus we have

$$F(x) = \sum_{o}^{k-1} f(j) \frac{(xq)^j}{(x+1)_{j+1}} + \frac{x^k}{(x+1)_k} G(xq^{-k}) ,$$

or

(5.72) $$\Lambda_q[E^k f;x] = x^{-k}(x+q^{-k})_k\{F(q^k x)-q^{-k} \sum_{o}^{k-1} \frac{(xq)^j}{(x+q^{-k})_{j+1}}\} .$$

The $(\Lambda_q,(-1)^n)$ transform transforms a linear difference equation with constant coefficients in $q^{-n}\Delta$ to an algebraic equation. Note that any such equation is also a linear q-difference equation with constant coefficients. This is so, since by the change of variables $q^{-n} = x$, and $f(n) = g(x)$, $q^{-n}\Delta f$ becomes $(q-1) D_q g(x)$. The inverse transform transforms a linear equation with constant coefficients

in $xD_q(1+x)$, say

$$(5.73) \qquad \sum_{k=o}^{j} a_k \{xD_q(x+1)\}^k F(x) = G(x) \quad ,$$

to $\{\sum_{k=o}^{j} a_k[n]^k\}f(n) = g(n)$, with $f(n) = \Lambda_q^{-1}[F;n]$ and $g(n) = \Lambda_q^{-1}[G;n]$. Consequently the solution to (5.73) is

$$F(x) = \Lambda_q\left[\frac{\Lambda_q^{-1}[G;n]}{\sum_{k=o}^{n} a_k[n]^k} ; x\right] .$$

Our final result in this section is to establish expansion formulas of powers of a general $A_{\alpha,q}$ operator in terms of powers of the shift operator E. This is a q-analogue of our investigations in §5.1. On the other hand the results of §5.1 may be looked at as the limiting case $q \to 1$ of our present result. Let

$$(5.74) \qquad A^{\ell}_{\alpha,q} f(n) = \sum_k (-1)^{\ell+n} a_{\ell,k}(n) E^k f(n) \quad .$$

In particular, if $f(n) = \theta_j(n)$, we get the relation

$$c_j c_{j-1} \cdots c_{j-\ell} \theta_{j-\ell}(n) = \sum_k a_{\ell,k}(n) \theta_j(n+k) \quad ,$$

or its equivalent

$$(-1)^{j+\ell+r} q^{(j-n-r)(j-n-r-1)/2} c_j c_{j-1} \cdots c_{j-\ell}$$
$$\times q^{-\ell j+\ell(\ell-1)/2} \left[{n \atop j-\ell}\right]\left[{j \atop n+r}\right]$$
$$= \sum_k a_{\ell,k}(n)\left[{n+k \atop j}\right]\left[{j \atop n+r}\right](-1)^{j+n+r} q^{(j-n-r)(j-n-r-1)/2} .$$

Summing the above equations for different j, with $n+r \leq j \leq n+k$ and using (5.32) we get

$$a_{\ell,r}(n) = \sum_{j} (-1)^{j+\ +r} q^{(j-n-r)(j-n-r-1)/2} \begin{bmatrix} n \\ j-\ell \end{bmatrix} \begin{bmatrix} j \\ n+r \end{bmatrix} \tag{5.75}$$

$$\times \ c_j \, c_{j-1} \cdots c_{j-\ell} \, q^{\ell(\ell-j-1)/2} \ .$$

Clearly (5.10) and (5.12) are the limiting case $q \to 1$ of (5.74) and (5.75).

5.3 Multidimensional (L,α) Transforms. A two dimensional (L,α) transform, say $L[f;\alpha_{m,n},x,y]$ can be defined as

$$L[f;\alpha_{m,n},x,y] = \sum_{o,o}^{\infty,\infty} \frac{(-x)^m(-y)^n}{m!\ n!} \frac{\partial^{m+n}\ \phi(x,y)}{\partial x^m\ \partial y^n} f_{m,n} \ , \tag{5.76}$$

on double sequences $\{f_{m,n}\}_{(o,o)}^{(\infty,\infty)}$, where $\phi(x,y)$ is the power series

$$\phi(x,y) = \sum_{o,o}^{\infty,\infty} \alpha_{m,n}\, x^m y^n \ . \tag{5.77}$$

We shall use the notation

$$\Delta_1 f(m,n) = f(m+1,n) - f(m,n)$$

$$\Delta_2 f(m,n) = f(m,n+1) - f(m,n) \ ,$$

and similar notation for E_1, E_2, ∇_1 and ∇_2.

The convolution product is defined by

$$f*g(m,n) = \sum_{r,s} (-1)^{i+j} \binom{m}{i}\binom{n}{j} \lambda_{i,j} \ , \tag{5.78}$$

where

$$(5.79) \qquad \lambda_{i,j} = \frac{1}{\alpha_{i,j}} \sum_{r=o}^{i} \sum_{u=o}^{j} \alpha_{i-r,j-s}\, \alpha_{r,s} \sum_{a,b;c,d} (-1)^{a+b+c+d}$$

$$\times \binom{r}{c}\binom{s}{d}\binom{i-r}{a}\binom{j-s}{d} f(a,b)\, g(c,d) \quad .$$

Equivalently,

$$(5.80) \qquad \alpha_{i,j}\, \Delta_1^i\, \Delta_2^j\, f{*}g(0,0) = \sum_{r=o}^{i} \sum_{u=o}^{j} \alpha_{r,s}\, \alpha_{i-r,j-s}$$

$$\times \quad \Delta_1^r\, \Delta_2^s\, f(0,0)\, \Delta_1^{i-r}\, \Delta_2^{j-s}\, g(0,0) \ .$$

Note that (5.76) may be written, formally, as

$$(5.81) \qquad L[f;\alpha,x,y] = \sum_{o,o}^{\infty} (-x)^n(-y)^m\, \alpha_{m,n}\, \Delta_1^m\, \Delta_2^n\, f(0,0) \quad ,$$

where α now stands for the double sequence $\{\alpha_{m,n}\}_{(o,o)}^{(\infty,\infty)}$.

We shall restrict ourselves to the case $\phi(x) = (1+x+y)^{-1}$, that is $\alpha_{i,j} = (-1)^{i+j}\binom{i+j}{i}$ and the general case can be developed on the same lines.

Easy manipulations lead to

$$(5.82) \qquad L[mf(m,n);(-1)^{m+n}\binom{m+n}{n},x,y] = x[1+y\,\frac{\partial}{\partial y} + (1+x)\,\frac{\partial}{\partial x}]F(x,y) \quad ,$$

and similarly, or by symmetry,

$$(5.83) \qquad L[nf(m,n);(-1)^{m+n}\binom{m+n}{m},x,y] = y[1+x\,\frac{\partial}{\partial x} + (1+x)\,\frac{\partial}{\partial y}]F(x,y) \quad ,$$

where

$$F(x,y) = L[f;(-1)^{m+n}\binom{m+n}{n},x,y] \quad .$$

Set

$$(5.84) \qquad \Omega = 1 + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} \quad .$$

One can show that

$$(5.85) \qquad L[(m+n+1)f(m,n);(-1)^{m+n}\binom{m+n}{m},x,y] = (1+x+y)\ F(x,y) \quad ,$$

$$(5.86) \qquad L[(m+n+1)E_1 f(m,n),(-1)^{m+n}\binom{m+n}{n},x,y] = (1+x+y)(\Omega + \frac{\partial}{\partial y})F(x,y)$$

$$(5.87) \qquad L[(m+n+1)E_2 f(m,n),(-1)^{m+n}\binom{m+n}{n},x,y] = (1+x+y)(\Omega + \frac{\partial}{\partial x})F(x,y) \quad ,$$

and

$$(5.88) \qquad L\left[\frac{\binom{m}{k}\binom{n}{j}}{\binom{k+j}{k}};\ \binom{m+n}{m}(-1)^{m+n},x,y\right] = x^k y^j \quad ,$$

for fixed k and j, $k,j = 0,1,\ldots$.

In this case we have two finite difference analogues of the Bessel operator. Indeed $\Delta_1(m\nabla_1+n\nabla_2)f(m,n)$ and $\Delta_2(m\nabla_1+n\nabla_2)f(m,n)$ are such operators since

$$(5.89) \qquad L[\Delta_1(m\nabla_1+n\nabla_2)f(m,n);(-1)^{m+n}\binom{m+n}{n},x,y] = \frac{\partial}{\partial x}F(x,y) \quad ,$$

and

$$(5.90) \qquad L[\Delta_2(m\nabla_1+n\nabla_2)f(m,n);(-1)^{m+n}\binom{m+n}{n},x,y] = \frac{\partial}{\partial y}F(x,y) \quad .$$

Furthermore we can prove the following formulas

$$(5.91) \qquad L[1+m\nabla_1+n\nabla_2)f(m,n);(-1)^{m+n}\binom{m+n}{n},x,y] = \Omega F(x,y) \quad ,$$

$$(5.92) \qquad L[\prod_{r=o}^{k}(r+m\nabla_1+n\nabla_2)\Delta_1^k f;(-1)^{m+n}\binom{m+n}{n},x,y] = \frac{\partial^k}{\partial x^k}F(x,y) \quad ,$$

and

$$(5.93)\qquad \Delta_1^k \prod_{r=o}^{k} (-r+m\nabla_1+n\nabla_2) = \prod_{r=o}^{k} (m\nabla_1+n\nabla_2+r)\Delta_1^k \quad ,$$

and two more formulas similar to (5.92) and (5.93) where Δ_1^k and $\frac{\partial^k}{\partial x^k}$ are replaced by Δ_2^k and $\frac{\partial^k}{\partial y^k}$. In particular one can conclude that the Laplace operator $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the transform under $(L,(-1)^{m+n} \binom{m+n}{n})$ to $(\Delta_1^2 + \Delta_2^2)(m\nabla_1+n\nabla_2-1)(m\nabla_1+n\nabla_2)$.

The general $(L,\alpha_{m,n})$ transforms may provide a simple method to solve difference equations in two variables. However, the present topic is far from being complete and will have to be investigated, in more details, in future.

We would like to point out that a combination of (L,α_n) and Carson-Laplace transforms may provide a very effective technique to solve functional equations. For example the integro-differential difference equation

$$(5.94)\qquad f(m+1,\eta)-f(m,\eta) + \frac{\partial f(m,\eta)}{\partial \eta} + \frac{d}{d\eta} \int_o^\eta k(m,\eta-t)f(m,t)dt = g(m,\eta) \quad ,$$

is transformed under T,

$$(5.95)\qquad T[f;x,y] = y^{-1} \int_o^\infty e^{-t/y} \sum_o^\infty \frac{x^n}{(1+x)^{n+1}} f(n,t)dt \quad ,$$

to

$$\frac{F(x,y)-F(0,y)}{x} + \frac{F(x,y)-F(x,0)}{y} + K(x,y)F(x,y) = G(x,y) \quad ,$$

where F, G and K are the T-transforms of f, g and k respectively. Therefore

$$(5.96) \qquad F(x,y) = \frac{xyG(x,y)+xF(x,0)+yF(0,y)}{xyK(x,y)+y+x} .$$

Now in order to get $f(m,\eta)$ form (5.96), one has to be able to invert the transform (5.95). Such inversion does not present any problem since the inverse is a composition of the inverses of the Carson-Laplace transform and the L transform.

In general one might be able to handle functional equations in several variables by using a different transform for each variable. The extension of (L,α) transforms from two to higher dimensions is rather obvious from (5.56).

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Table of the L-Transform

$f = \{f(n)\}_o^\infty$	$F(x) = L[f;x]$
$\{\binom{n}{j}\}_{n=o}^\infty$, $j = 0,1,\ldots$	x^j , $j = 0,1,\ldots$
$\{\nabla \sum_o^n g(k)f(n-k)\}_{n=o}^\infty$	$G(x)F(x)$
Δf	$\frac{F(x)-F(0)}{x}$
$\Delta^k f$, $k = 1,2,\ldots$	$\frac{F(x)-F(0)-xF'(0)-\ldots-x^{k-1}F^{(k-1)}(0)}{x^k}$, $k = 1,2,\ldots$
∇f	$F(x)/(1+x)$
$\nabla^k f$, $k = 1,2,\ldots$	$\frac{F(x)}{(1+x)^k}$, $k = 1,2,\ldots$
$\beta f = \{\Delta n \nabla f(n)\}_o^\infty$	$f'(x)$
$E^{-1}f$	$(\frac{x}{x+1})\ F(x)$
Ef	$(\frac{x+1}{x})\ F(x) - \frac{f(0)}{x}$
$\{nf(n)\}_o^\infty$	$xD\{(x+1)F(x)\}$
$\{(n+1)\Delta f(n)\}_o^\infty$	$(x+1)F'(x)$
$\{n^k f(n)\}_o^\infty$, $k = 1,2,\ldots$	$\{xD(1+x)\}^k\ F(x)$, $k = 1,2,\ldots$
$\{n^{(k)}\nabla^k f(n)\}_{n=o}^\infty$, $k = 1,2,\ldots$	$x^k D^k F(x)$, $k = 1,2,\ldots$

$f = \{f(n)\}_o^\infty$	$F(x) = L[f;x]$
$\{\sum_o^n f(k)\}_{n=o}^\infty$	$(x+1)\ F(x)$
$\{\Delta^\ell[n^k \nabla^k f(n)]\}_{n=o}^\infty$, $k=1,2,\ldots$	$x^{k-\ell}\ D^k F(x)$
$\{a^n f(n)\}_{n=o}^\infty$	$\frac{1}{1+x-ax}\ F(\frac{ax}{1+x-ax})$
$e^{\lambda\beta}\ f$	$F(x+\lambda)$
$1 = (1,1,1,\ldots)$	1
$(1,0,0,\ldots)$	$\frac{1}{1+x}$
$(0,1,1,\ldots)$	$\frac{x}{x+1}$
$(-1)^n \binom{k}{n}$, $k = 0,1,\ldots$	$(1+x)^{-k-1}$, $k = 0,1,\ldots$
$(1,1,0,0,0,\ldots)$	$1 - (\frac{x}{x+1})^2$
$(\underbrace{1,1,\ldots,1}_{n\text{-times}},0,0,\ldots)$, $k=1,2,\ldots$	$1 - (\frac{x}{x+1})^k$, $k = 1,2,\ldots$
$\{\sum_{k=o}^n f(k)g(n-k)\}_{n=o}^\infty$	$(x+1)F(x)G(x)$
$(1,2,3,\ldots)$	$x+1$
$(0,1,2,\ldots)$	x
$\{\binom{k+n}{m}\}_{n=o}^\infty$, $k,m = 0,1,\ldots;$ $k \geq m$	$x^{m-k}(1+x)^k$, $k,m = 0,1,\ldots;$ $k \geq m$

$f = \{f(n)\}_{o}^{\infty}$	$F(x) = L;x]$
$\{1,2,2,\ldots\}$	$\frac{2x+1}{x+1} = 1 + \frac{x}{x+1}$
$\{\binom{n+\alpha}{n}\}_{n=o}^{\infty}$	$(1+x)^{\alpha}$
$\{{}_1F_1(-n;1;t)\}_{n=o}^{\infty}$	e^{-xt}
$\{\cos nt\}_{n=o}^{\infty}$	$\frac{1}{2x+1}$
$\{\sin \frac{\pi}{2} n\}_{n=o}^{\infty}$	$\frac{x}{1+2x+2x^2}$
$\{\cos n \frac{\pi}{2}\}_{n=o}^{\infty}$	$\frac{x+1}{2x^2+2x+1}$

Some Operational Formulas for Berg's Calculus

$$(f*g)(n) = \nabla \sum_{o}^{n} f(k)g(n-k)$$

$$\{\delta_{n,o}\}_{n=o}^{\infty} * f = \{\nabla f\}$$

$$\{1-\delta_{n,o}\}_{n=o}^{\infty} * f = \{E^{-1} f\}$$

$$\{n\}*\{\Delta f\} = f - \{f(0)\}$$

$$\{\delta_{n,i}\}_{n=o}^{\infty} * f = \{E^{-i} \nabla f\}$$

$$\{n+1\}*f = \{\sum_{o}^{n} f_k\}_{o}^{\infty} = \sigma f$$

$$\{n\}_{o}^{\infty} * f = \{n\nabla f(n)\}_{o}^{\infty}$$

$$\beta(f*g) = f*(\beta g) + g*(\beta f)$$

$$\{n\}_{o}^{\infty} * \beta f + \{\frac{n(n-1)}{2}\}_{o}^{\infty} * \beta f = \{nf(n-1)\}$$

$$\{\frac{n(n+1)}{2}\}_{o}^{\infty} * \beta f + \{n\}_{o}^{\infty} * f = \{nf(n)\}$$

$$(1,1,0,0,\ldots)*f = \{f(n)-f(n-2)\}$$

$$(\underbrace{1,1,\ldots,1}_{n\text{-times}},0,0,\ldots)*f = \{f(n)-f(n-k)\} \quad , \quad k = 1,2,\ldots$$

$$\{(-1)^n \binom{\ell}{n}\}*f = \{\sum_{o}^{r} g(k)(-1)^{n-k} \binom{\ell-1}{n-k}\} \quad , \quad \ell = 1,2,\ldots$$

$$(1,2,2,\ldots)*f = \{f(n) + f(n-1)\}$$